Acta Numerica (2002), pp. 73–144
DOI: 10.1017/S0962492902000028

Subdivision schemes in geometric modelling

Nira Dyn and David Levin
School of Mathematical Sciences,
Tel Aviv University,
Tel Aviv 69978, Israel
E-mail: `niradyn@post.tau.ac.il`
`levin@post.tau.ac.il`

Subdivision schemes are efficient computational methods for the design and representation of 3D surfaces of arbitrary topology. They are also a tool for the generation of refinable functions, which are instrumental in the construction of wavelets. This paper presents various flavours of subdivision, seasoned by the personal viewpoint of the authors, which is mainly concerned with geometric modelling. Our starting point is the general setting of scalar multivariate nonstationary schemes on regular grids. We also briefly review other classes of schemes, such as schemes on general nets, matrix schemes, non-uniform schemes and nonlinear schemes. Different representations of subdivision schemes, and several tools for the analysis of convergence, smoothness and approximation order are discussed, followed by explanatory examples.

CONTENTS

1. Introduction

The first work on a subdivision scheme was by de Rahm (1956). He showed that the scheme he presented produces limit functions with a first derivative everywhere and a second derivative nowhere. The pioneering work of Chaikin (1974) introduced subdivision as a practical algorithm for curve design. His algorithm served as a starting point for extensions into subdivision algorithms generating any spline functions. The importance of subdivision to applications in computer-aided geometric design became clear with the generalizations of the tensor product spline rules to control nets of arbitrary topology. This important step has been introduced in two papers, by Doo and Sabin (1978) and by Catmull and Clark (1978). The surfaces generated by their subdivision schemes are no longer restricted to representing bivariate functions, and they can easily represent surfaces of arbitrary topology.

In recent years the subject of subdivision has gained popularity because of many new applications, such as 3D computer graphics, and its close relationship to wavelet analysis. Subdivision algorithms are ideally suited to computer applications: they are simple to grasp, easy to implement, highly flexible, and very attractive to users and to researchers. In free-form surface design applications, such as in the 3D animation industry, subdivision methods are already in extensive use, and the next venture is to introduce these methods to the more conservative, and more demanding, world of geometric modelling in the industry.

Important steps in subdivision analysis have been made in the last two decades, and the subject has expanded into new directions owing to various generalizations and applications. This review does not claim to cover all aspects of subdivision schemes, their analysis and their applications. It is, rather, a personal view of the authors on the subject. For example, the convergence analysis is not presented in its greatest generality and is restricted to uniform convergence, which is relevant to geometric modelling. On the other hand, the review deals with the analysis and applications of nonstationary subdivision schemes, which the authors view as important for future developments. While most of the analysis presented deals with convergence and regularity, it also relates the results to practical issues such as attaining optimal approximation order and computing limit values.

The presentation starts with the basic notions of nonstationary subdivision: definitions of limit functions and basic limit functions and the refinement relations they satisfy. Different forms of representation of subdivision schemes, and their basic convolution property, are also presented in Section 2. These are later used throughout the review for stating and proving the main results. In the next section we present a gallery of examples of different types of subdivision schemes: interpolatory and non-interpolatory, linear and nonlinear, stationary and nonstationary, matrix subdivision, Hermite-type subdivision, and bivariate subdivision on regular and irregular nets. In the same section we also sketch some extensions of subdivision schemes that are not studied in this review. The material in Sections 2 and 3 is intended to provide a broad map of the subdivision area for tourists and new potential users.

In Section 4, the convergence analysis of univariate and bivariate stationary subdivision schemes, and the smoothness analysis of their limit functions, are presented via the related difference schemes. Analogous analysis is also presented for nonstationary schemes, relating the results to the analysis of stationary subdivision and using smoothing factors and convolutions as main tools. The central results are given, some with full proofs and others with only sketches. The special analysis of convergence and smoothness at extraordinary points, of subdivision schemes on nets of general topology, is reviewed in Section 6. In Section 7 we discuss two issues in the practical application of subdivision schemes. One is the computation of exact limit values of the function (surface), and the limit derivatives, at dyadic points. The other is the approximation order of subdivision schemes, and how to attain it.

For other reviews and tutorials on subdivision schemes and their applications, the reader may turn to Cavaretta, Dahmen and Micchelli (1991), Schröder (2001), Zorin and Schröder (2000) and Warren (1995*b*)

2. Basic notions

This review presents subdivision schemes mainly as a tool for geometric modelling, starting from the general perspective of nonstationary schemes.

2.1. Nonstationary schemes

A subdivision scheme is defined as a set of refinement rules relative to a set of nested meshes of isolated points (nets)

$$N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq \mathbb{R}^s.$$

Each refinement rule maps real values defined on N_k to real values defined on a refined net N_{k+1}. The subdivision scheme is the repeated refinement of initial data defined on N_0 by these rules.

Let us first consider the regular grid case, namely the net $N_0 = \mathbb{Z}^s$ for $s \in \mathbb{Z}_+\backslash 0$ and its binary refinements, namely the refined nets $N_k = 2^{-k}\mathbb{Z}^s$, $k \in \mathbb{Z}_+\backslash 0$. Let $\mathbf{f}^k$ be the values attached to the net $N_k = 2^{-k}\mathbb{Z}^s$,

$$\mathbf{f}^k = \{f^k_\alpha : \alpha \in \mathbb{Z}^s\} \tag{2.1}$$

with f^k_α attached to $2^{-k}\alpha$.

The refinement rule at refinement level k is of the form

$$f_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta}^k f_\beta^k, \quad \alpha\in\mathbb{Z}^s, \tag{2.2}$$

which we write formally as

$$\mathbf{f}^{k+1} = R_{\mathbf{a}^k}\mathbf{f}^k. \tag{2.3}$$

The set of coefficients $\mathbf{a}^k = \{a_\alpha^k : \alpha\in\mathbb{Z}^s\}$ determines the refinement rule at level k and is termed the kth *level mask*. Let $\sigma(\mathbf{a}^k) = \{\gamma \mid a_\gamma^k \neq 0\}$ be the support of the mask $\mathbf{a}^k$. Here we restrict the discussion to the case that the origin is in the convex hull of $\sigma(\mathbf{a}^k)$, and that $\sigma(\mathbf{a}^k)$ are finite sets, for $k\in\mathbb{Z}_+$. A more general form of refinement, corresponding to a dilation matrix M, is

$$f_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-M\beta}^k f_\beta^k, \tag{2.4}$$

where M is an $s\times s$ matrix of integers with $|\det(M)|>1$ (see, *e.g.*, Dahmen and Micchelli (1997) and Han and Jia (1998)). In this case the refined nets are $M^{-k}\mathbb{Z}^s$, $\;k\in\mathbb{Z}_+$. We restrict our discussion to binary refinements corresponding to $M=2I$, with I the $s\times s$ identity matrix, namely to (2.2).

If the masks $\{\mathbf{a}^k\}$ are independent of the refinement level, namely if $\mathbf{a}^k = \mathbf{a}$, $k\in\mathbb{Z}_+$, the subdivision scheme is termed *stationary*, and is denoted by $S_\mathbf{a}$. In the nonstationary case, the subdivision scheme is determined by $\{\mathbf{a}^k : k\in\mathbb{Z}_+\}$, and is denoted as a collection of refinement rules $\{R_{\mathbf{a}^k}\}$, or by the shortened notation $S_{\{\mathbf{a}^k\}}$.

2.2. Notions of convergence

A continuous function $f\in C(\mathbb{R}^s)$ is termed the *limit function* of the subdivision scheme $S_{\{\mathbf{a}^k\}}$, from the initial data sequence $\mathbf{f}^0$, and is denoted by $S_{\{\mathbf{a}^k\}}^\infty\mathbf{f}^0$, if

$$\lim_{k\to\infty}\max_{\alpha\in\mathbb{Z}^s\cap K} |f_a^k - f(2^{-k}\alpha)| = 0, \tag{2.5}$$

where $\mathbf{f}^k$ is defined recursively by (2.2), and K is any compact set in $\mathbb{R}^s$.

This is equivalent (Cavaretta *et al.* 1991) to f being the uniform limit on compact sets of $\mathbb{R}^s$ of the sequence $\{F_k : k\in\mathbb{Z}_+\}$ of s-linear spline functions interpolating the data at each refinement level, namely

$$F_k(2^{-k}\alpha) = f_\alpha^k, \quad F_k\big|_{2^{-k}(\alpha+[0,1]^s)} \in \pi_1^T, \quad \alpha\in\mathbb{Z}^s, \tag{2.6}$$

where π_1^T is the tensor product space of the spaces of linear polynomials in each of the variables.

From this equivalence we get

$$\lim_{k\to\infty} \|f(t) - F_k(t)\|_{\infty,K} = 0. \tag{2.7}$$

If we do not insist on the continuity of f in (2.5) or on the L_∞-norm in (2.7), we get weaker notions of convergence: for instance, L_p-convergence is defined by requiring the existence of $f \in L_p(\mathbb{R}^s)$ satisfying $\lim_{k\to\infty} \|f(t) - F_k(t)\|_p = 0$ (Villemoes 1994, Jia 1995). The case $p = 2$ is important in the theory of wavelets (Daubechies 1992). In this paper we consider mainly the notion of uniform convergence, corresponding to (2.7), which is relevant to geometric modelling. We also mention here the weakest notion of convergence (Derfel, Dyn and Levin 1995), termed *weak convergence* or *distributional convergence*. A subdivision scheme $S_{\{\mathbf{a}^k\}}$, generating the values $f^{k+1} = R_{a^k} f^k$, for $k \in \mathbb{Z}_+$, converges weakly to an integrable function f if, for any $g \in C_0^\infty$ (infinitely smooth and of compact support),

$$\lim_{k\to\infty} 2^{-k} \sum_{\alpha\in\mathbb{Z}^s} g(2^{-k}\alpha) f_\alpha^k = \int_{\mathbb{R}}^s f(x)g(x)\,\mathrm{d}x.$$

Definition 1. A subdivision scheme is termed *uniformly convergent* if, for any initial data, there exists a limit function in the sense of (2.7) (or equivalently, if, for any initial data, there exists a continuous limit function in the sense of (2.5)) and if the limit function is nontrivial for at least one initial data sequence. A uniformly convergent subdivision scheme is termed C^m, or C^m*-convergent* if, for any initial data, the limit function has continuous derivatives up to order m.

In the following we use the term convergence for uniform convergence, since this notion of convergence is central to the review.

An important initial data sequence is $\mathbf{f}^0 = \boldsymbol{\delta} = \{f_\alpha^0 = \delta_{\alpha,0} : \alpha \in \mathbb{Z}^s\}$. If $S_{\{\mathbf{a}^k\}}$ is convergent, then there exists a nontrivial limit function starting from this initial data sequence:

$$\phi_{\{\mathbf{a}^k\}} = S_{\{\mathbf{a}^k\}}^\infty \boldsymbol{\delta}.$$

By the uniformity of the refinement rules (each refinement rule operates in the same way at all locations), and by their linearity,

$$S_{\{\mathbf{a}^k\}}^\infty \mathbf{f}^0 = \sum_{\alpha\in\mathbb{Z}^s} f_\alpha^0 \phi_{\{\mathbf{a}^k\}}(\cdot - \alpha), \tag{2.8}$$

for any initial $\mathbf{f}^0$. Thus, if $\phi_{\mathbf{a}^k} \in C^m(\mathbb{R}^s)$ for some $m \geq 0$, so is any limit function generated by $S_{\mathbf{a}^k}$, and the scheme is C^m.

When the initial data consist of a sequence of vectors

$$\mathbf{P}^0 = \{P_\alpha^0 \in \mathbb{R}^d : \alpha \in \mathbb{Z}^s\} \in (\ell_\infty(\mathbb{Z}^s))^d,$$

the limit of the subdivision, given by (2.8) with $\mathbf{f}^0$ replaced by $\mathbf{P}^0$, is a parametric representation of a manifold in $\mathbb{R}^d$. In geometric modelling $s = 1$ corresponds to curves in $\mathbb{R}^d$ for $d = 2, 3$ and $s = 2, d = 3$ to surfaces in $\mathbb{R}^3$. The set of refined points $\mathbf{P}^k$, for $k \in \mathbb{Z}_+$, is termed the *control points at level* k.

2.3. The refinement equations

The function $\phi_{\{\mathbf{a}^k\}} = S^\infty_{\{\mathbf{a}^k\}}\boldsymbol{\delta}$, termed the *basic limit function* of the subdivision scheme $S_{\{\mathbf{a}^k\}}$, is the first in the family of functions $\{\phi_\ell : \ell \in \mathbb{Z}_+\}$, defined by

$$\phi_\ell = S^\infty_\ell \boldsymbol{\delta}, \tag{2.9}$$

where $S_\ell = \{R_{\mathbf{a}^k} : k \geq \ell,\ k \in \mathbb{Z}_+\}$. Each function in this family is a basic limit function of a subdivision scheme defined in terms of a subset of the masks $\{\mathbf{a}^k\}$. If $S_0 = S_{\{\mathbf{a}^k\}}$ is convergent so is any S_ℓ for $\ell \in \mathbb{Z}_+$ (Dyn and Levin 1995) (see Section 4.1). Thus all the functions $\{\phi_\ell : \ell \in \mathbb{Z}_+\}$ are well defined, if S_0 is convergent. Moreover, by (2.9),

$$S^\infty_\ell \mathbf{f}^0 = \sum_{\alpha \in \mathbb{Z}^s} f^0_\alpha \phi_\ell(\cdot - \alpha). \tag{2.10}$$

The support of ϕ_ℓ can be determined by the the supports of the masks $\{\mathbf{a}^k\}$. Recalling that $\sigma(\mathbf{a}^k)$ denotes the support of the mask $\mathbf{a}^k$, which is a finite set of points in $\mathbb{Z}^s$, then, by the refinement rules (2.2) and by (2.9), the support $\sigma(\phi_\ell)$ of ϕ_ℓ is given by

$$\sigma(\phi_\ell) = \overline{\sum_{k=\ell}^{\infty} 2^{\ell-k-1}\sigma(\mathbf{a}^k)}, \tag{2.11}$$

where the sum denotes the Minkowski sum of sets (that is, $A + B = \{a + b : a \in A,\ b \in B\}$). In the stationary case and in the univariate case, (2.11) can be further elaborated.

In the univariate case, $s = 1$, let $[\ell^k, u^k] = \langle \sigma(\mathbf{a}^k) \rangle$ be the convex hull of $\sigma(\mathbf{a}^k)$, and let

$$\ell_k = \sum_{j=k}^{\infty} 2^{k-j-1}\ell^j, \qquad u_k = \sum_{j=k}^{\infty} 2^{k-j-1}u^j.$$

Then

$$\sigma(\phi_k) \subseteq [\ell_k, u_k]. \tag{2.12}$$

In the stationary case (Cavaretta *et al.* 1991), (2.11) yields

$$\sigma(\phi_{\mathbf{a}}) \subseteq \langle \sigma(\mathbf{a}) \rangle. \tag{2.13}$$

The functions $\{\phi_k : k \in \mathbb{Z}_+\}$ are related by a system of functional equations, termed *refinement equations*. To see this, observe that $(R_{\mathbf{a}^k}\boldsymbol{\delta})_\alpha = a^k_\alpha$,

$\alpha \in \mathbb{Z}^s$ and, by the linearity of the refinement rules,

$$\phi_k = \sum_{\alpha} a_{\alpha}^k \phi_{k+1}(2 \cdot - \alpha), \quad k \in \mathbb{Z}_+. \tag{2.14}$$

In the stationary case, namely when $\mathbf{a}^k = \mathbf{a},\ k \in \mathbb{Z}_+$, this system of equations reduces to a single functional equation

$$\phi_{\mathbf{a}} = \sum_{\alpha} a_{\alpha} \phi_{\mathbf{a}}(2 \cdot - \alpha), \tag{2.15}$$

with $\mathbf{a} = \{a_\alpha : \alpha \in \mathbb{Z}^s\}$, and $\phi_{\mathbf{a}} = S_{\mathbf{a}}^{\infty} \boldsymbol{\delta}$.

The refinement equation (2.15) is the key to the generation of multiresolution analysis and wavelets (Daubechies 1992, Mallat 1989). When the scheme $S_{\mathbf{a}}$ converges, the unique compactly supported solution of the refinement equation (2.15) coincides with $S_{\mathbf{a}}^{\infty} \boldsymbol{\delta}$. The refinement equation (2.15) suggests another way to compute its unique compactly supported solution. This method is termed the 'cascade algorithm' (see, *e.g.*, Daubechies and Lagarias (1992*a*)). It involves repeated use of the operator

$$T_{\mathbf{a}} g = \sum_{\alpha} a_{\alpha} g(2 \cdot - \alpha).$$

defined on continuous compactly supported functions. The cascade algorithm is as follows.

(1) Choose a continuous compactly supported function, ψ_0, as a 'good' initial guess (*e.g.*, H as in (2.20)).

(2) Iterate: $\psi_{k+1} = T_{\mathbf{a}} \psi_k$.

It is easy to verify that the operator $T_{\mathbf{a}}$ is the adjoint of the refinement rule $R_{\mathbf{a}}$, in the following sense: for any ψ continuous and of compact support, (Cavaretta *et al.* 1991)

$$\sum_{\alpha} (R_{\mathbf{a}} f)_{\alpha} \psi(2 \cdot - \alpha) = \sum_{\alpha} f_{\alpha} (T_{\mathbf{a}} \psi)(\cdot - \alpha). \tag{2.16}$$

Note that, while the refinement rule $R_{\mathbf{a}}$ is defined on sequences, the operator $T_{\mathbf{a}}$ is defined on functions. A similar operator to $T_{\mathbf{a}}$, defined on sequences, is

$$(\widetilde{T}_{\mathbf{a}} \mathbf{f})_{\alpha} = \sum_{\beta} a_{\beta} f_{2\alpha - \beta} = \sum_{\gamma} a_{2\alpha - \gamma} f_{\gamma}. \tag{2.17}$$

This operator is the adjoint of the operator $R_{\mathbf{a}}$ on the space of sequences defined on $\mathbb{Z}^s$. The operator $\widetilde{T}_{\mathbf{a}}$ in (2.17) is termed the *transfer operator* (Daubechies 1992), and plays a major role in the analysis of the solutions of refinement equations of the form (2.15) (see, *e.g.*, Jia (1996), Han (2001), Han and Jia (1998) and Jia and Zhang (1999)).

2.4. Representations of subdivision schemes

The notions introduced above regard a subdivision scheme $S_{\{\mathbf{a}^k\}} = \{R_{\mathbf{a}^k}\}$ as a set of operators defined on sequences in $\ell_\infty(\mathbb{Z}^s)$. Each refinement rule can be represented as a bi-infinite matrix with each element indexed by two index vectors from $\mathbb{Z}^s$,

$$f_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} A_{\alpha,\beta}^k f_\beta^k, \quad \alpha \in \mathbb{Z}^s, \tag{2.18}$$

where the bi-infinite matrix A^k has elements

$$A_{\alpha,\beta}^k = a_{\alpha-2\beta}^k. \tag{2.19}$$

Finite sections of these matrices are used in the analysis of the subdivision scheme $S_{\{\mathbf{a}^k\}}$ (see Section 5).

One may also regard a subdivision scheme as a set of operators $\{R_k : k \in \mathbb{Z}_+\}$ defined on a function space (Dyn and Levin 1995), if one considers the functions $\{F_k\}$ introduced in (2.6). The set of operators $\{R_k\}$ has the property that R_k maps F_k into F_{k+1}. More specifically, let H be defined by

$$H(\alpha) = \boldsymbol{\delta}_{0,\alpha}, \quad H\big|_{(\alpha+[0,1]^s)} \in \pi_1^T, \quad \alpha \in \mathbb{Z}^s. \tag{2.20}$$

Define the operators $\{R_k\}$ on $C(\mathbb{R}^s)$ as

$$R_k g = \sum_{\alpha\in\mathbb{Z}^s} H(2^{k+1}\cdot -\alpha) \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta}^k g(2^{-k}\beta), \quad k \in \mathbb{Z}_+, \tag{2.21}$$

for any $g \in C(\mathbb{R}^s)$. Then the subdivision scheme $S_{\{\mathbf{a}^k\}}$ is related to the set of operators $\{R_k\}$ in several ways, for example,

$$(R_k g)\big|_{2^{-k-1}\mathbb{Z}^s} = R_{\mathbf{a}^k}(g\big|_{2^{-k}\mathbb{Z}^s}),$$

and the more significant relation

$$S_{\{\mathbf{a}^k\}}^\infty \mathbf{f}^0 = \lim_{k\to\infty} R_k R_{k-1} \cdots R_0 g, \tag{2.22}$$

where $g \in C(\mathbb{R}^s)$ is any interpolant to $\mathbf{f}^0$ on $\mathbb{Z}^s$, namely

$$g(\alpha) = f_\alpha^0, \quad \alpha \in \mathbb{Z}^s.$$

In particular, g can be

$$g = \sum_{\alpha\in\mathbb{Z}^s} H(\cdot - \alpha) f_\alpha^0.$$

Another important relation is

$$\|R_k\| = \|R_{\mathbf{a}^k}\|_\infty = \max_{\alpha\in E_s} \left\{ \sum_{\beta\in\mathbb{Z}^s} |a_{\alpha-2\beta}^k| \right\}, \tag{2.23}$$

where E_s is the set of extreme points of $[0,1]^s$. The representation of subdivision schemes in terms of sequences of operators on $C(\mathbb{R}^s)$ facilitates the application of standard operator-theory tools to the analysis of subdivision schemes, for instance, to deduce convergence properties of nonstationary schemes from those of related stationary ones (Dyn and Levin 1995) (see Section 4.1).

A representation of the refinement rule (2.2), which is a central tool in the convergence and smoothness analysis of stationary schemes, is in terms of z-transforms (Laurent series). Let the symbol of the mask $\mathbf{a}^k$ be defined as the Laurent polynomial

$$a^k(z) = \sum_{\alpha \in \mathbb{Z}^s} a^k_\alpha z^\alpha. \tag{2.24}$$

Here we use the multi-index notation $z^n = z_1^{n_1} \cdots z_s^{n_s}$, for $z \in \mathbb{R}^s,\ n \in \mathbb{Z}^s$, and $z^n = z_1^n \cdots z_s^n$, for $z \in \mathbb{R}^s,\ n \in \mathbb{Z}$. Obviously, a subdivision scheme $S_{\{\mathbf{a}^k\}}$ is identified with the set of its symbols $\{a^k(z)\}$. In our notation we exchange freely between the mask and its symbol, for instance, $\phi_{\{a^k(z)\}}$ denotes the basic limit function of $S_{\{a^k(z)\}} = S_{\{\mathbf{a}^k\}}$.

Let the z-transform of the sequence $\mathbf{f} = \{f_\alpha : \alpha \in \mathbb{Z}^s\}$ be denoted by $L(\mathbf{f}; z)$, namely

$$L(\mathbf{f}; z) = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha z^\alpha.$$

Then the refinement rule (2.2) can be written in the form

$$L(\mathbf{f}^{k+1}; z) = a^k(z) L(\mathbf{f}^k; z^2), \tag{2.25}$$

with the formal meaning of the equality above being that corresponding powers of z on both sides of the equality have equal coefficients. Iterating the relation (2.25), we obtain

$$L(\mathbf{f}^{k+\ell}; z) = a^{k+\ell-1}(z) a^{k+\ell-2}(z^2) \cdots a^k(z^{2^{\ell-1}}) L(\mathbf{f}^k; z^{2^\ell}). \tag{2.26}$$

Thus, the ℓ-iterated symbol from level k to level $k+\ell$ is

$$a^{[k;\ell]}(z) = \sum_{\alpha \in \mathbb{Z}^s} a^{[k;\ell]}_\alpha z^\alpha = \prod_{j=1}^{\ell} a^{k+\ell-j}(z^{2^{j-1}}). \tag{2.27}$$

In the stationary case we denote the ℓ-iterated symbol by $a^{[\ell]}$:

$$a^{[\ell]}(z) = \prod_{j=1}^{\ell} a(z^{2^{j-1}}). \tag{2.28}$$

2.5. The convolution property

Here we present an important property of schemes, which is easily expressed in terms of the Laurent polynomial representation. This property is presented in three different forms, depending on the notion of convergence used.

(1) Let $S_{\{\mathbf{a}^k\}}$ and $S_{\{\mathbf{b}^k\}}$ each be either (uniformly) convergent or convergent in the sense of (2.5), with corresponding basic limit functions $\phi_{\{\mathbf{a}^k\}}$ and $\phi_{\{\mathbf{b}^k\}}$ continuous in their support. Then the scheme $S_{\{\mathbf{c}^k\}}$ defined by the symbols

$$c^k(z) = 2^{-s} a^k(z) b^k(z) \tag{2.29}$$

is also convergent, and its basic limit function is

$$\phi_{\{\mathbf{c}^\mathbf{k}\}} = \phi_{\{\mathbf{a}^k\}} * \phi_{\{\mathbf{b}^\mathbf{k}\}}. \tag{2.30}$$

Here the symbol $*$ stands for the s-dimensional convolution (Cavaretta *et al.* 1991, Dyn and Levin 1995).

The convolution property which is repeatedly used in this paper for $s > 1$, is of a different form.

(2) Let $S_{\{\mathbf{b}^\mathbf{k}\}}$ be a convergent s-variate subdivision scheme, and let $S_{\{\mathbf{a}^k\}}$ be a univariate scheme, which is convergent in the sense of (2.5) to integrable limit functions. Then the symbols

$$c^k(z) = 2^{-1} a^k(z^\lambda) b^k(z), \tag{2.31}$$

with $\lambda \in \mathbb{Z}^s$, define a convergent scheme $S_{\{\mathbf{c}^\mathbf{k}\}}$. Moreover,

$$\phi_{\{\mathbf{c}^k\}}(x) = \phi_{\{\mathbf{a}^k\}} *_\lambda \phi_{\{\mathbf{b}^k\}}(x) \equiv \int_{\mathbb{R}} \phi_{\{\mathbf{a}^k\}}(x - \lambda t) \phi_{\{\mathbf{b}^k\}}(t)\, \mathrm{d}t. \tag{2.32}$$

The convolution property is also valid in the case of weak convergence of $S_\mathbf{a}$. This property is used in only one example in the paper.

(3) Let $S_{\{\mathbf{b}^k\}}$ be an s-variate subdivision scheme convergent in the sense of (2.5), with $\phi_{\{\mathbf{b}^k\}}$ continuous in its support, and let $S_{\{\mathbf{a}^k\}}$ be a weakly convergent s-variate scheme, with $\phi_{\{\mathbf{a}^k\}}$ continuous in its support. Then the scheme $S_{\{\mathbf{c}^\mathbf{k}\}}$ defined by the symbols in (2.29) is convergent, and $\phi_{\{\mathbf{c}^\mathbf{k}\}}$ is given by (2.30).

Here we indicate how to verify convolution property (2) ((2.31) and (2.32)). The verification of the convolution property in its other two forms is based on the same reasoning. Observe that, for $\mathbf{f}^k = R_{\mathbf{a}^{k-1}} \cdots R_{\mathbf{a}^0} \boldsymbol{\delta}$, we have $L(\mathbf{f}^k; z) = a^{[0,\ell]}(z)$, and that in polynomial multiplication the coefficients are computed by convolutions of the coefficients of the factors. Thus, the relations (2.31) and (2.27) yield

$$c^{[0,\ell]}(z) = 2^{-\ell} a^{[0,\ell]}(z^\lambda) b^{[0,\ell]}(z),$$

or equivalently

$$L(\mathbf{g}^\ell; z) = 2^{-\ell} L(\mathbf{f}^\ell; z^\lambda) L(\mathbf{h}^\ell; z), \tag{2.33}$$

with $\mathbf{g}^\ell = R_{\mathbf{c}^{\ell-1}} \cdots R_{\mathbf{c}^0}\boldsymbol{\delta}$, and $\mathbf{h}^\ell = R_{\mathbf{b}^{\ell-1}} \cdots R_{\mathbf{b}^0}\boldsymbol{\delta}$.

Now, (2.32) can be concluded by equating coefficients of equal powers of z on both sides of (2.33), taking into account the convergence of $\{\mathbf{f}^k\}_{k\in\mathbb{Z}_+}$ and of $\{\mathbf{h}^k\}_{k\in\mathbb{Z}_+}$ to the compactly supported limit functions $\phi_{\{\mathbf{a}^k\}}$ and $\phi_{\{\mathbf{b}^k\}}$, respectively.

3. The variety of subdivision schemes

Subdivision schemes were first studied as a tool for generating spline functions (Chaikin 1974, Riesenfeld 1975, Cohen, Lyche and Riesenfeld 1980). The renewed interest in this subject in geometric modelling has evolved as subdivision processes were extended to general topologies (Catmull and Clark 1978, Doo and Sabin 1978). In recent years interesting applications have emerged, such as wavelet theory, and some very challenging theoretical issues have arisen. In the following we discuss the major different types of subdivision schemes, most of them relevant to geometric modelling:

- B-spline and box-spline schemes
- the up-function scheme
- exponential spline and exponential box-spline schemes
- interpolatory schemes
- shape-preserving schemes
- general matrix schemes
- Hermite-type and moment interpolatory schemes
- tensor product schemes
- different topologies for surface subdivision.

While assessing the various types we incorporate the notions of local support and support size, smoothness and approximation order. These issues will be further developed and investigated in the next sections. Here we take the liberty of using these properties in a heuristic manner.

3.1. Elementary schemes and their convolutions

The simplest elementary univariate uniform stationary scheme is the scheme defined by the symbol

$$a^k(z) = a(z) = 1 + z. \tag{3.1}$$

The corresponding basic limit function is the characteristic function of $[0, 1]$, where the convergence is in the sense of (2.5):

$$\phi_{\mathbf{1+z}} = B_0(\cdot) = \chi_{[0,1]}. \tag{3.2}$$

By convolution property (1),

$$\phi_{2^{-m}(1+z)^{m+1}} = B_0(\cdot) * B_0(\cdot) * \cdots * B_0(\cdot) = B_m(\cdot). \tag{3.3}$$

Thus, the scheme with symbol $a(z) = 2^{-m}(1+z)^{m+1}$ has as a basic limit function the mth-degree B-spline function with integer knots, supported in $[0, m+1]$, which is in $C^{m-1}(\mathbb{R})$. As shown in Section 4.2, the symbol of a C^m univariate uniform stationary binary scheme, under an additional mild condition, must contain the factor $(1+z)^{m+1}$. The earliest example of a spline subdivision is Chaikin's algorithm (Chaikin 1974)

$$f_{2i}^{k+1} = \frac{3}{4}f_i^k + \frac{1}{4}f_{i+1}^k, \qquad f_{2i+1}^{k+1} = \frac{1}{4}f_i^k + \frac{3}{4}f_{i+1}^k, \tag{3.4}$$

which converges to the quadratic spline $\sum f_i^0 B_2(\cdot - i)$. Chaikin's algorithm is also the basic example of a 'corner cutting' algorithm, which served as a starting point to various generalizations, for instance in de Boor (1987) and Gregory and Qu (1996). The application of three iterations on a simple control polygon (the polygonal line joining the control points) is presented in Figure 3.1.

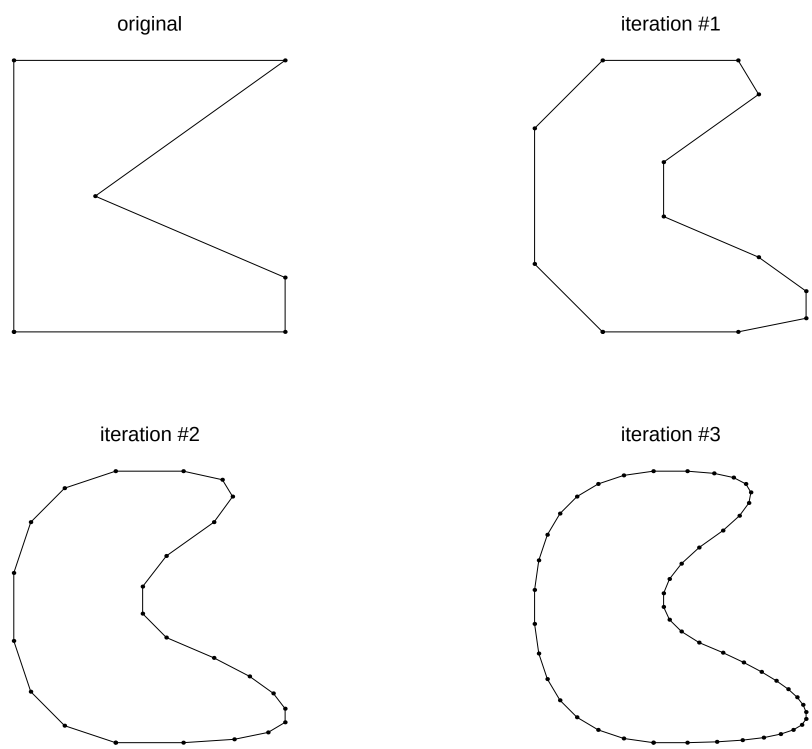

Figure 3.1. Chaikin's algorithm

Another interesting scheme that is constructed by convolutions of the elementary scheme is defined by

$$a^k(z) = 2^{-k+1}(1+z)^k. \tag{3.5}$$

The corresponding basic limit function is Rvachev's up-function (Rvachev 1990, Derfel *et al.* 1995) which is in $C^\infty(\mathbb{R})$ and is supported in $[0,2]$ (see Example 5). The spaces $V_k = \operatorname{span}\{\phi_k(2^k \cdot -\alpha) : \alpha \in \mathbb{Z}^s\}$, $k \in \mathbb{Z}_+$, with $\{\phi_k\}$ defined as (2.9) with respect to the symbols at (3.5), provide spectral approximation order (Dyn and Ron 1995).

Products of the elementary univariate factors in directions in $\mathbb{Z}^s$ generate box-splines in $\mathbb{R}^s$ as basis limit functions. Let $\Lambda = \{\lambda_1, \ldots, \lambda_\ell\} \subset \mathbb{Z}^s$, and define the stationary scheme with the symbol

$$a(z) = 2^{s-\ell} \prod_{j=1}^{\ell} (1 + z^{\lambda_j}). \tag{3.6}$$

This scheme is related to the box-spline when directions Λ (de Boor, Höllig and Riemenschneider 1993, Dahmen and Micchelli 1984). Convergence is guaranteed if there is a subset of s directions $\{\lambda_{i_1}\lambda_{i_2}, \ldots, \lambda_{i_s}\} \in \Lambda$ such that $\det(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}) = 1$. Furthermore, if any subset of $\ell - m - 1$ directions spans $\mathbb{R}^s$, then $\phi_{\mathbf{a}}$ is in C^m (de Boor *et al.* 1993).

An important example here is the scheme generating the C^2 quartic 3-directional box-spline, namely, the scheme with the symbol

$$a(z) = 2^{-4}(1+z^{(1,0)})^2(1+z^{(0,1)})^2(1+z^{(1,1)})^2. \tag{3.7}$$

It is easy to check that the above conditions are satisfied with $m = 2$, and thus the basic limit function is a box-spline in C^2.

The uniform nonstationary elementary scheme is again a scheme defined by symbols that are linear polynomials in z, namely

$$a^k(z) = 1 + r_k z, \quad k \in \mathbb{Z}_+. \tag{3.8}$$

The parameters $\{r_k\}_{k\in\mathbb{Z}_+}$ are free parameters which determine the convergence of the subdivision scheme and the regularity of the limit function (Dyn and Levin 1995). To examine these issues we write the scheme explicitly as

$$f_{2i}^{k+1} = f_i^k, \quad f_{2i+1}^{k+1} = r_k f_i^k, \quad i \in \mathbb{Z}. \tag{3.9}$$

Starting the subdivision with initial data sequence $\mathbf{f}^0 = \boldsymbol{\delta}$, the limit at a dyadic point $x = \sum_{i=1}^k d_i 2^{-i} \in [0,1)$, $d_i \in \{0,1\}$, is determined at level k of the subdivision. It is easy to verify that the value of the basic limit function ϕ at a dyadic point is given by

$$\phi(x) = \prod_{i=1}^{k} r_{i-1}^{d_i}. \tag{3.10}$$

Let us define $\phi(x)$ at nondyadic points by

$$\phi(x) = \prod_{i=1}^{\infty} r_{i-1}^{d_i}, \qquad x = \sum_{i=1}^{\infty} d_i 2^{-i} \in [0,1), \tag{3.11}$$

and $\phi(x) = 0$ for all $x \notin [0,1)$. If we assume that the parameters $\{r_k\}$ satisfy $\sum_{k\in\mathbb{Z}_+} |1 - r_k| < \infty$, then all the above infinite products converge, and we find out that ϕ is continuous at all nondyadic points. At dyadic points in $[0,1)$ ϕ is right-continuous; hence, it is integrable. As proved in Dyn and Levin (1995), ϕ is also left-continuous at all dyadic points in $(0,1)$ if and only if $r_k = e^{c2^{-k}}$ for some constant c.

Exponential B-splines. The univariate elementary nonstationary scheme defined by

$$a^k(z) = 1 + e^{c2^{-k-1}} z, \quad k \in \mathbb{Z}_+, \tag{3.12}$$

generates the exponential B-spline

$$\phi_{\{\mathbf{a^k}\}}(x) = e^{cx} \chi_{[0,1]}(x). \tag{3.13}$$

Consequently, by convolution property (1), the scheme generating the mth-order exponential B-spline with exponents $c_1, \dots, c_m$ is

$$a^k(z) = 2^{-m+1} \prod_{j=1}^{m} (1 + e^{c_j 2^{-k-1}} z). \tag{3.14}$$

Similarly, one can derive symbols of schemes generating exponential box-splines and exponential up-functions (Dyn and Levin 1995).

Generating circumscribed circles. A special example of a scheme that is obtained by convolution of elementary schemes is given by the symbol

$$a^k(z) = \frac{1}{2(1+\cos(\alpha_k))}(1+z)(1+e^{i\alpha_k}z)(1+e^{-i\alpha_k}z),$$
$$\alpha_k = 2^{-k-1}\alpha_0, \quad k \in \mathbb{Z}_+. \tag{3.15}$$

This is a C^1 'corner cutting' scheme which reproduces constants and also $\sin(\alpha_0 x)$, $\cos(\alpha_0 x)$. If the initial control polygon is a regular n-gon and $\alpha_0 = 2\pi/n$, then the limit curve is the circle circumscribed in the n-gon (Dyn and Levin 1992). The tensor product of the above scheme with any other stationary scheme generates surfaces of revolution (Morin, Warren and Weimer 2001). It seems that a circle cannot be generated by a linear stationary scheme.

3.2. Interpolatory schemes

A class of subdivision schemes with many specific features is that of 'interpolatory subdivision schemes' (Dyn and Levin 1990). The schemes in this class generate the refined values by retaining the values at the vertices of the current net, and defining new values at the new vertices of the refined net.

Among the B-spline schemes, only those generating B_0 and B_1 are interpolatory schemes, that is, satisfying

$$f_{2j}^{k+1} = f_j^k, \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}_+, \tag{3.16}$$

together with insertion rules for new points $\{f_{2j+1}^{k+1}\}_{j\in\mathbb{Z}}$. The interpolatory refinement rules on $N_k = 2^{-k}\mathbb{Z}^s$ have the form

$$f_{2\alpha}^{k+1} = f_\alpha^k, \quad f_{\gamma+2\alpha}^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\gamma+2\beta}^k f_{\alpha-\beta}^k, \quad \gamma \in E_s\backslash 0, \quad \alpha \in \mathbb{Z}^s. \tag{3.17}$$

The masks corresponding to an interpolatory subdivision scheme have the feature

$$a_{2\alpha}^k = \delta_{\alpha,0}, \quad \alpha \in \mathbb{Z}^s, \quad k \in \mathbb{Z}_+.$$

It is easy to realize that, in case of a convergent scheme, all the points

$$(2^{-k}\alpha, f_\alpha^k), \quad \alpha \in \mathbb{Z}^s, \quad k \in \mathbb{Z}_+,$$

are on the graph of the limit function. In this setting there is (uniform) convergence if the values generated at the dyadic points $\{f_\alpha^k : \alpha \in \mathbb{Z}^s,\ k \in \mathbb{Z}_+\}$ are continuous.

The basic limit functions $\{\phi_k : k \in \mathbb{Z}_+\}$ satisfy

$$\phi_k(\alpha) = \delta_{\alpha,0}, \quad \alpha \in \mathbb{Z}^s, \quad k \in \mathbb{Z}_+,$$

thus their integer shifts $\{\phi_k(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$ are linearly independent for any $k \in \mathbb{Z}_+$.

The following examples are univariate stationary schemes. Nonstationary univariate interpolatory schemes are discussed in Example 2. A bivariate interpolatory scheme is presented in Section 3.5.

The 4-point scheme. The first stationary interpolatory schemes were the 4-point schemes presented in Dubuc (1986) and Dyn, Gregory and Levin (1987). The 4-point scheme is the univariate scheme defined by (3.16) and the insertion rule

$$f_{2j+1}^{k+1} = -wf_{j-1}^k + \left(\frac{1}{2} + w\right) f_j^k + \left(\frac{1}{2} + w\right) f_{j+1}^k - wf_{j+2}^k, \tag{3.18}$$

for $j \in \mathbb{Z}$, and $k \in \mathbb{Z}_+$, where w is a shape parameter of the scheme. The symbol of the 4-point scheme is

$$a_w(z) = \frac{1}{2z}(z+1)^2(1 + wb(z)), \tag{3.19}$$

where

$$b(z) = -2z^{-2}(z-1)^2(z^2+1). \tag{3.20}$$

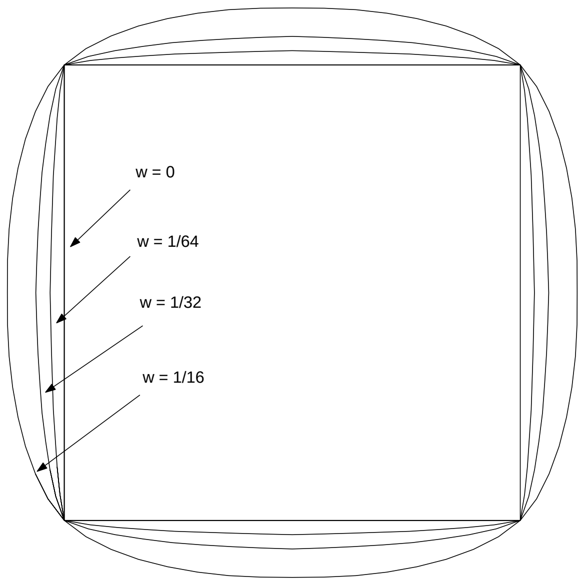

Figure 3.2. Curves generated by the 4-point scheme

For $w = 0$ the limit is the piecewise linear interpolant to the data. As w increases, the limit function is less tight. The symbol contains the elementary factor $(z+1)^2$ necessary for C^1 convergence, and the challenge in Dyn *et al.* (1987) was to determine the range of values of the shape parameter w for which the scheme is C^1. The particular value $w = \frac{1}{16}$ is also analysed in Dubuc (1986). In this case the symbol contains the factor $(z+1)^4$, which means that the scheme reproduces cubic polynomials (see Section 4.2). Yet the limit function is not even C^2. It is shown in Dyn, Gregory and Levin (1991) that the 4-point scheme is C^1 for any $w \in (0, \frac{\sqrt{5}-1}{8})$, and in Deslauriers and Dubuc (1989) that, for $w = \frac{1}{16}$, the first derivative is Hölder-continuous for any Hölder exponent $0 < \nu < 1$, yet the second derivative does not exist at dyadic points (Dyn *et al.* 1987). The application of four iterations of the 4-point scheme, with different shape parameters, on a square control polygon is presented in Figure 3.2.

Dubuc–Deslauriers interpolatory schemes. The 4-point scheme of Dubuc (1986) has been generalized to symmetric $2n$-point interpolatory schemes by Deslauriers and Dubuc (1989). The insertion rule for f^{k+1}_{2j+1} is defined by the value of the interpolation polynomial of degree $2n-1$ at $2^{-k-1}(2j+1)$, interpolating the $2n$ values $f^k_{j-n+1}, \ldots, f^k_{j+n}$. Let us denote the resulting symbol by $d_{(2n)}(z)$. These schemes are studied in Deslauriers and Dubuc (1989) by Fourier analysis, and their convergence is proved. The smoothness of $S_{\mathbf{d}_{(2n)}}$ grows linearly, but slowly, with n (Daubechies 1992).

Generalizations to multidimensional interpolatory schemes are presented in Dyn, Gregory and Levin (1990*a*) and Riemenschneider and Shen (1997).

In analogy to the up-function, it is possible to get C_0^∞ interpolatory basic limit functions using the symbols of Dubuc–Deslauriers interpolatory schemes. This is achieved in Cohen and Dyn (1996) by defining the nonstationary subdivision symbols as $a^k(z) = d_{(2k)}(z)$.

Nonlinear, shape-preserving 4-point schemes. A significant drawback of linear interpolatory schemes is the lack of shape-preservation properties. If one is interested in both interpolation and shape preservation, then linearity has to be given up. A beautiful example of a nonlinear, stationary, shape-preserving interpolatory scheme is the following 4-point C^1 convexity-preserving scheme due to Kuijt and van Damme (1998), where the rule replacing (3.18) is

$$f_{2j+1}^{k+1} = \frac{1}{2}(f_j^k + f_{j+1}^k) - \frac{1}{4\left(\frac{1}{d_j^k} + \frac{1}{d_{j+1}^k}\right)}; \quad d_j^k = f_{j+1}^k - 2f_j^k + f_{j-1}^k. \tag{3.21}$$

Starting with strictly convex initial functional data, it is shown in Kuijt and van Damme (1998) that the limit function is a strictly convex C^1 function. Kuijt and van Damme (1999) have also developed nonlinear schemes preserving monotonicity. It is also possible to use the linear 4-point scheme and to generate a convex limit function from initial strictly convex data, by choosing $w \in (0, w^*)$, where w^* depends on the initial data (Dyn, Kuijt, Levin and van Damme 1999*a*).

3.3. Matrix schemes and Hermite-type schemes

While interpolatory schemes preserve function data at points of the previous level, it is sometimes desirable to preserve other quantities. Two related families of schemes of this kind are Hermite-type schemes and moment-interpolating schemes. We may view interpolatory schemes as schemes generating limit functions with specified values at the integers. Hermite-type schemes generate limit functions with specified function values and certain derivatives' values at the integers. Moment-interpolating schemes produce limit functions with specified moments on the intervals $[i, i+1]$, $i \in \mathbb{Z}$. In both cases, the data attached to the vertices of the nets form a vector of values, and the subdivision operator is defined by a mask with matrix elements.

A univariate uniform stationary matrix subdivision scheme, operating on sequences of vectors in $\mathbb{R}^n$, is defined by a set of real $n \times n$ matrix coefficients $\{A_j : j \in \mathbb{Z}\}$, with a finite number of nonzero A_js, generating sequences of control points in $\mathbb{R}^n$, $\mathbf{v}^k = \{v_j^k \in \mathbb{R}^n : j \in \mathbb{Z}\}$, $k \geq 0$, recursively, by

$$v_i^{k+1} = \sum_{j \in \mathbb{Z}} A_{i-2j} v_j^k, \quad i \in \mathbb{Z}. \tag{3.22}$$

As an example of such a scheme, we consider the scheme generating the double-knot cubic splines. The matrix mask is defined by its matrix symbol,

$$A(z) = \frac{1}{16}\begin{pmatrix} 2+6z+z^2 & 2z+5z^2 \\ 5+2z & 1+6z+2z^2 \end{pmatrix} = \sum_{i\in\mathbb{Z}} A_i z^i. \tag{3.23}$$

Here there are two basic sets of initial data, namely $\mathbf{v}^{1,0} = (1,0)^t\boldsymbol{\delta}$ and $\mathbf{v}^{2,0} = (0,1)^t\boldsymbol{\delta}$. The two basic limit vector functions are

$$S^\infty_{\mathbf{A}}\mathbf{v}^{1,0} = (\phi_1,\phi_1)^t, \qquad S^\infty_{\mathbf{A}}\mathbf{v}^{2,0} = (\phi_2,\phi_2)^t, \tag{3.24}$$

where ϕ_1 and ϕ_2 are the two different cubic B-splines spanning the space of cubic splines with double knots at the integers (Plonka 1997).

Let us now return to the Hermite-type and moment-interpolating schemes. In the Hermite case we start with Hermite-type data, $\{v^0_j = (f^0_j, g^0_j)^t\}_{j\in\mathbb{Z}}$ where the values $\{g^0_j\}$ represent derivative data. We now consider the scheme

$$v^{k+1}_{2i} = v^k_v, \quad v^{k+1}_{2i+1} = \sum A^{(k)}_{1-2j} v^k_{i+j}, \quad k \ge 0, \tag{3.25}$$

or, equivalently,

$$v^{k+1}_i = \sum_j A^{(k)}_{i-2j} v^k_j, \quad k \ge 0, \tag{3.26}$$

where $\{A^{(k)}_i\}$ are 2×2 matrices, possibly depending upon the refinement level k, and $A^{(k)}_{2j} = \delta_{j,0} I_{2\times 2}$. The Hermite-type scheme recursively defines values $\{v^k_j = (f^k_j, g^k_j)^t\}_{j\in\mathbb{Z}}$ attached respectively to the dyadic points $\{j2^{-k}\}_{j\in\mathbb{Z}}$. We say that the scheme is C^r if there exists a function $f \in C^r(\mathbb{R})$ such that

$$v^k_j = (f^k_j, g^k_j)^t = (f(j2^{-k}), f'(j2^{-k}))^t, \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}_+. \tag{3.27}$$

The first interesting example, presented in Merrien (1992), is an extension of the interpolatory Hermite-cubic rule. The nonzero matrices of its mask are

$$A^{(k)}_1 = \begin{pmatrix} \frac{1}{2} & \alpha 2^{-k} \\ -\beta 2^k & \frac{1-\beta}{2} \end{pmatrix}, \qquad A^{(k)}_{-1} = \begin{pmatrix} \frac{1}{2} & -\alpha 2^{-k} \\ \beta 2^k & \frac{1-\beta}{2} \end{pmatrix}. \tag{3.28}$$

This scheme with $\alpha = 1/8$ and $\beta = 3/2$ produces the piecewise Hermite-cubic interpolant to the given initial data, and thus it is a C^1-scheme. We note that the matrices depend upon k, and they are even unbounded as $k \to \infty$. However, as shown in Dyn and Levin (1999), if we consider in this case the scheme for transforming the vector of values $u^k_j = (g^k_j, df^k_j)^t$, with $df^k_j = 2^k(f^k_j - f^k_{j-1})$, this scheme becomes stationary, that is, with a constant matrix mask. Here, if the original scheme is C^1, then both elements of $\{u^k_j\}$ should converge to the same limit function f'.

The moment interpolation problem for m moments is defined as follows. Let $b^\ell(x) = \frac{(m-1)!}{\ell!(m-1-\ell)!}x^l(1-x)^{m-1-l}\cdot\chi_{[0,1]}$ denotes the ℓth Bernstein polynomial of degree $m-1$ for the interval $[0,1]$, truncated to $[0,1]$. Define

$$b_j^\ell(x) = b^\ell(x-j),$$

the translate of b^ℓ that 'lives' on $[j, j+1]$ and has L_1-norm 1.

Given the local moments of a function f,

$$\beta_j^\ell = \langle f, b_j^\ell \rangle, \quad j \in \mathbb{Z}, \quad 0 \le \ell < m, \tag{3.29}$$

the problem is to construct a 'smooth' function $\tilde{f}$ matching those moments. A solution of this problem by a subdivision process is presented in Donoho, Dyn, Levin and Yu (2000). Also shown there is the close relationship between the moment-interpolating subdivision schemes and the Hermite interpolatory subdivision schemes. In the sections on the analysis of subdivision schemes, we consider only schemes with scalar masks. The analysis of schemes with a matrix mask is not reviewed here. The interested reader may consult Plonka (1997), Cohen, Daubechies and Plonka (1997), Cohen, Dyn and Levin (1996), Micchelli and Sauer (1998) and Dyn and Levin (2002).

3.4. Tensor product schemes and related ones

The simplest subdivision schemes on $\mathbb{Z}^2$ are the stationary tensor product schemes, obtained by applying one stationary univariate scheme in the x-direction and then a second (or the same) stationary univariate scheme in the y-direction. Let us denote the symbols of the stationary univariate schemes by $x(z)$ and $y(z)$, respectively; then the symbol of the tensor product scheme S_t is $t(z_1, z_2) = x(z_1)y(z_2)$. Obviously, the tensor product subdivision scheme inherits the convergence and smoothness properties of the univariate schemes. Tensor products of univariate spline schemes are special cases of box-splines, using only two directions in (3.6). For example, the mask generating the biquadratic and the bicubic B-spline functions are, respectively, defined by the symbols

$$a(z_1, z_2) = 2^{-4}(1+z_1)^3(1+z_2)^3, \tag{3.30}$$

$$a(z_1, z_2) = 2^{-6}(1+z_1)^4(1+z_2)^4. \tag{3.31}$$

Yet tensor product schemes have masks of relatively large support for given smoothness. For box-splines, the same smoothness may be achieved by using more directions in (3.6), and fewer linear factors (see Section 4.3).

Considering the case of interpolatory schemes, the tensor product of two 4-point schemes (3.18) has the mask $t_w(z_1.z_2) = a_w(z_1)a_w(z_2)$, with an insertion rule based on 16 points. Yet, as shown in Dyn, Hed and Levin (1993), an interpolatory scheme with insertion rule of smaller support size (12 points) and with the same polynomial precision and smoothness exists.

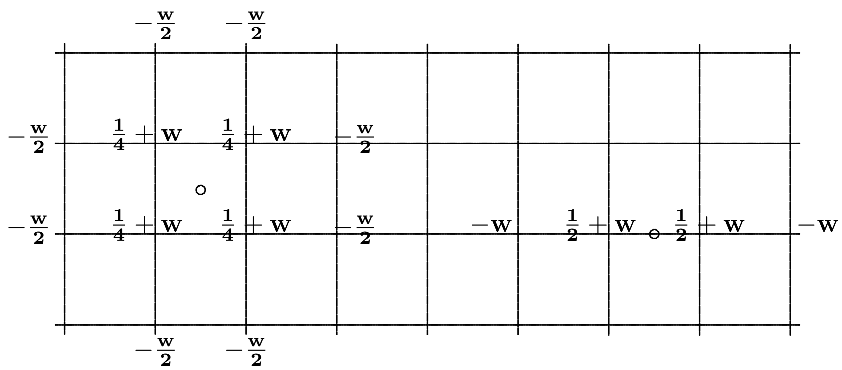

Figure 3.3. The two stencils of the truncated tensor product scheme S_{c_w}

The suggested scheme is obtained by removing all the w^2 terms in t_w. The resulting symbol is

$$c_w(z_1, z_2) = \frac{1}{4}(1+z_1)^2(1+z_2)^2 z_1^{-1} z_2^{-1}(1 - w[b(z_1) + b(z_2)]), \qquad (3.32)$$

where b is given in (3.20).

The stencils (see Section 3.5) of the insertion rule of this truncated tensor product scheme are shown in Figure 3.3.

The scheme S_{c_w} reproduces cubic polynomials for $w = \frac{1}{16}$, and it reduces to the 4-point scheme in one direction, when the data values are constant along the other direction (Dyn *et al.* 1993). An interpolatory subdivision on quadrilateral nets (see Section 3.5), with arbitrary topology based on the 4-point scheme, is proposed by Kobbelt (1996*a*).

3.5. Subdivision on nets

We consider control nets for generating surfaces in $\mathbb{R}^3$, a control net consists of control points in $\mathbb{R}^3$ with topological relations between them. The refinement rules are defined with respect to a control net, and generate a refined control net with new control points. The topological relations in the refined net are determined by the type of net, while the control points are determined by the subdivision scheme as weighted averages of topologically neighbouring control points.

In this section we present subdivision schemes that are defined over nets of arbitrary topology in 3D space. Such nets are valuable for the design of free-form surfaces. The surfaces generated by subdivision schemes on such nets are no longer restricted to bivariate functions, and they can represent surfaces of arbitrary topology. We describe three types of nets: triangular, Catmull–Clark type (primal type) and Doo–Sabin type (dual type), which are the most commonly used.

In addition to the above types of nets, there are hexagonal nets. Very few subdivision schemes with respect to hexagonal nets are available (see, *e.g.*, Dyn, Levin and Liu (1992) and Dyn, Levin and Simoens (2001*b*)), and they are not considered here.

Nets of general topology

A net $N(V, E, F)$, as shown in Figure 3.4, is a configuration of a finite set V of points in $\mathbb{R}^3$ called *vertices*, with two sets of topological relations between them E and F, called *edges* and *faces*. (A similar description of nets can be found in Kobbelt, Hesse, Prautzsch and Schweizerhof (1996).)

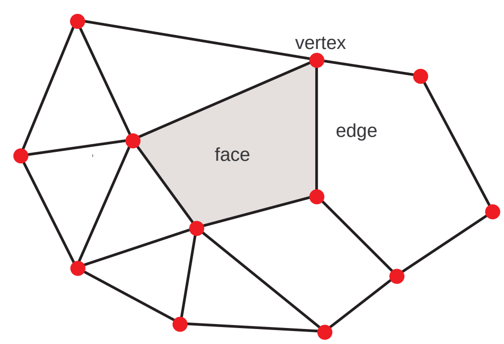

Figure 3.4. A net

An edge denotes a connection between two vertices. A face is a cyclic list of vertices where every pair of consecutive vertices shares an edge. The *valency* of a vertex, or a face, is the number of edges that share that vertex, or that face. While edges can always be represented by straight line segments, the vertices of a face are not necessarily co-planar; therefore a face is not associated with any geometric shape (in contrast to the faces of a polyhedron, which are planar pieces).

An edge e is called a *boundary edge* of $N(V, E, F)$ if it is not shared by two faces. A vertex v is called a *boundary vertex* if it belongs to a boundary edge.

We restrict our attention to nets $N(V, E, F)$ that satisfy the following properties:

(1) any two vertices share at most one edge;
(2) the valency of each vertex is at least 2.;
(3) the valency of each face is at least 3;
(4) every boundary edge belongs to exactly one face;
(5) three boundary edges cannot share a vertex.

$N(V, E, F)$ is said to be *closed* if it has no boundary edges. Otherwise, $N(V, E, F)$ is an *open* net. A triangular net is a net whose faces all have valency 3. A closed triangular net is termed *regular*, or a *regular triangulation*, if the valency of each vertex is 6. A regular triangular net is locally topologically equivalent to a portion of the 3-directional grid, that is, the grid $\mathbb{Z}^2$ with edges connecting (i, j) with $(i+1, j)$, $(i, j+1)$ and $(i+1, j+1)$, for $(i, j) \in \mathbb{Z}^2$. A quad-mesh is a net whose faces all have valency 4. A quad-mesh (quadrilateral net) is termed *regular* if it is topologically equivalent to $\mathbb{Z}^2$, that is, the valency of each vertex is 4.

The subdivision process transforms the net $N(V, E, F)$ into a refined net $N(V', E', F')$, where each new vertex in V' is associated with an element or a configuration c of elements from $N(V, E, F)$. The method for calculating a new vertex $v' \in V'$ can be described as a weighted average (with possibly negative weights) of vertices of V. The weight given to every vertex $v \in V$ depends only on its topological relation to c. The set of weights, together with their topological location in V relative to c, constitute the *stencil* which is determined by the subdivision scheme. There are different stencils for different topological configurations.

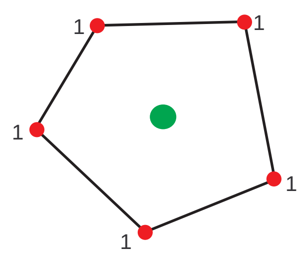

Figure 3.5. A stencil

For example, suppose that a vertex v' is associated with a face $f \in F$ that has valency 5. The stencil in Figure 3.5 represents the rule: v' is the average of the vertices of f. The set of vertices with nonzero weights, the support of the stencil, is topologically related to c, but does not necessarily coincide with c, as occurs in the last example. Together with the definition of V', there is a definition of the new edges E' and faces F', and these are described later for the different types of nets.

Let S denote a subdivision operator for nets. Let $N_0 = N(V, E, F)$ be a given initial net. A sequence of finer nets $N_k = N\left(V^k, E^k, F^k\right)$ is defined by

$$N_{k+1} = SN_k, \quad k = 0, 1, \ldots. \tag{3.33}$$

Ideally, the convergence of the sequence of nets $\{N_k : k \in \mathbb{Z}_+\}$ to a limit surface X should be defined independently of any parametrization of the surface. In the following definition, a surface X is considered as a closed subset of $\mathbb{R}^3$. We say that X is the limit surface of the subdivision scheme (3.33) if

$$\lim_{k\to\infty} \operatorname{dist}\bigl(V^k, X\bigr) = 0. \tag{3.34}$$

where $\operatorname{dist}(X,Y) = \max\{\sup_{y\in Y}\inf_{x\in X}\|x-y\|_2, \sup_{x\in X}\inf_{y\in Y}\|x-y\|_2\}$, is the Euclidean Hausdorff distance between two closed subsets $X, Y \subset \mathbb{R}^3$. When a limit surface X exists we denote it by $S^\infty N_0 = X$. In practice, however, the convergence is studied with respect to appropriate local parametrizations of the limit surface.

Triangular subdivision

Triangular subdivision schemes are defined over triangular nets, that is, nets whose faces all have valency 3 and therefore can be regarded as planar triangles. The new vertices are divided to v-vertices, and e-vertices. Each v-vertex in V' is associated with a vertex in V. Every e-vertex in V' is associated with an edge in E. For each type of vertex there is a different stencil. The new edges E' are defined between a new v-vertex and all the e-vertices such that their 'parents' in E share the parent of the v-vertex in V, and between any two e-vertices such that their parent edges share a face in F. Thus every triangle in the original net $N(V,E,F)$ is replaced by four triangles in the new net $N(V',E',F')$. The topology of the new triangular net is shown in Figure 3.6.

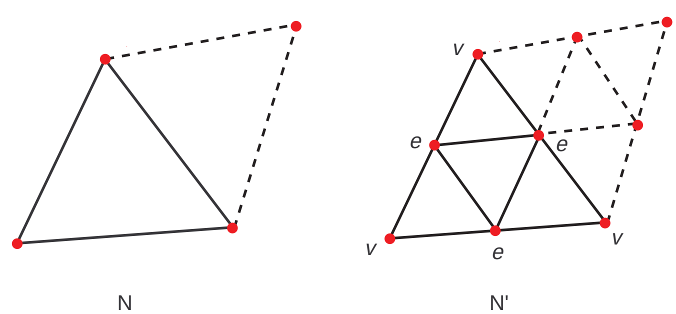

Figure 3.6. Triangular subdivision

A regular vertex in a triangular net is a vertex with valency 6. In a closed net, every new e-vertex has valency 6, and every new v-vertex inherits the valency of its parent vertex. Therefore, the number of irregular vertices in a net remains constant, and most of the net is a regular triangular net.

One of the commonly used triangular subdivision schemes is the Loop subdivision scheme (Loop 1987) defined for closed triangular nets. The stencils for the new e-vertices and v-vertices are depicted in Figure 3.7.

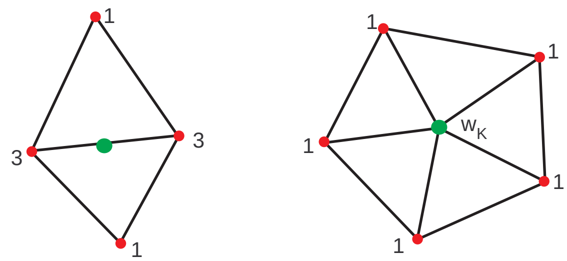

Figure 3.7. Loop scheme: stencils for e-vertex (left) and for v vertex (right)

The weight w_n given to the original vertex, in the stencil for its corresponding new v-vertex, depends on the valency K of that vertex. It is given by the following formula:

$$w_K = \frac{64K}{40 - \left(3 + 2\cos\left(\frac{2\pi}{K}\right)\right)^2} - K, \quad K = 3, 4, \ldots. \tag{3.35}$$

The Loop scheme generalizes the 3-directional box-spline scheme (3.7), in the sense that it coincides with it in the regular parts of the net. This implies that the limit surface is C^2 almost everywhere, and this is achieved with stencils of very small support. Near irregular vertices of the original net, the surface is C^1 (Loop 1987). Another property of this scheme, important for geometric modelling, is shape preservation, which is due to the positivity of the weights in the stencils of the Loop scheme.

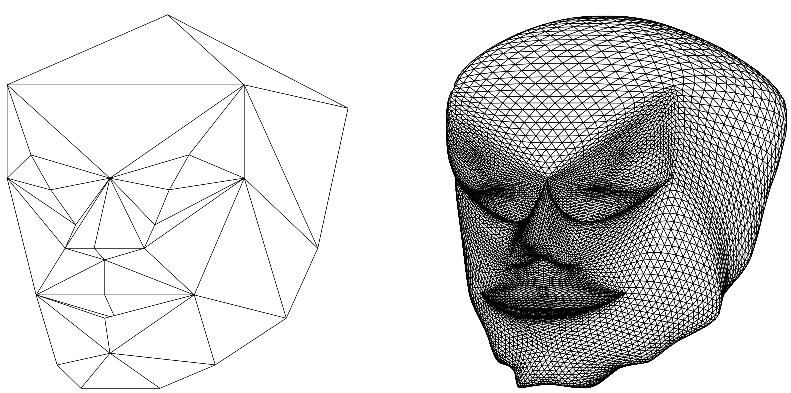

Figure 3.8. Head: initial control net (left), four butterfly iterations (right)

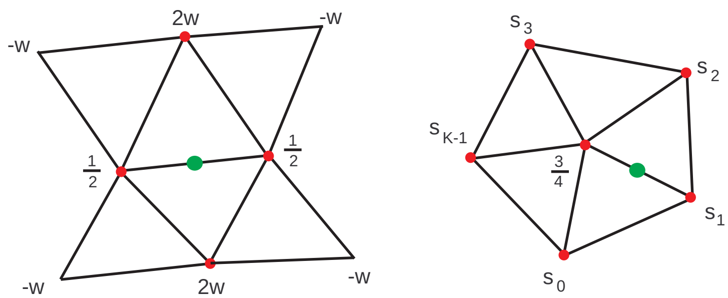

Figure 3.9. Modified Butterfly scheme: stencils corresponding to a 'regular' edge (left) and an 'irregular' edge (right)

An interpolatory triangular subdivision scheme with stencil of small support is the butterfly scheme (Dyn *et al.* 1990*a*). This scheme is defined over closed triangular nets. The application of four iterations of the butterfly insertion rule to an initial closed triangulation is depicted in Figure 3.8. There are modified stencils in the vicinity of irregular vertices (Zorin, Schröder and Sweldens 1996), which produce better-looking and smoother surfaces in the presence of irregular vertices.

As an interpolatory scheme, the new v-vertices inherit their location from their parent vertices. Figure 3.9 shows the stencils for new e-vertices. The butterfly stencil is used to calculate new e-vertices whose parent edge is 'regular', namely, has two regular vertices. A different stencil is used when the parent edge is 'irregular', namely, has one vertex which is regular and one which has valency $K \neq 6$. The weights $\{s_j\}_{j=0,\ldots,K-1}$ depend on the valency of the irregular vertex, and are given by

$$s_j = \frac{1}{K}\left(\tfrac{1}{4} + \cos\left(\tfrac{2\pi j}{K}\right) + \tfrac{1}{2}\cos\left(\tfrac{4\pi j}{K}\right)\right), \quad j = 0, \ldots, K-1.$$

The case where both of the vertices of the parent edge are irregular can occur only in the initial net. In such a case, in the first refinement step the calculation of the new e-vertex may be done in any reasonable way. The limit surfaces generated by the butterfly scheme are C^1 continuous everywhere, a property valuable for computer graphics applications (Zorin *et al.* 1996). An extended butterfly interpolatory subdivision scheme for the generation of C^2 surfaces on regular grids is presented in Labkovsky (1996).

Subdivision on an arbitrary net

The two types of refinements of nets of arbitrary topological structure are the Catmull–Clark type, also called 'primal', and the Doo–Sabin type, also called 'dual'.

In **primal-type refinement**, every face of valency n in the original net $N(V,E,F)$ is replaced by n quadrilateral faces in the new net $N'(V',E',F')$, as shown in Figure 3.10.

The new vertices are divided into v-vertices, e-vertices and f-vertices. Each v-vertex in V' is associated with a vertex in V. Each e-vertex in V' is associated with an edge in E. Each f-vertex in V' is associated with a face in F.

Figure 3.10 indicates the topological relations in $N(V',E',F')$, with the points v, e, and f indicating v-vertices, e-vertices and f-vertices, respectively. The new edges are marked by line segments and the faces by the quadrilaterals formed.

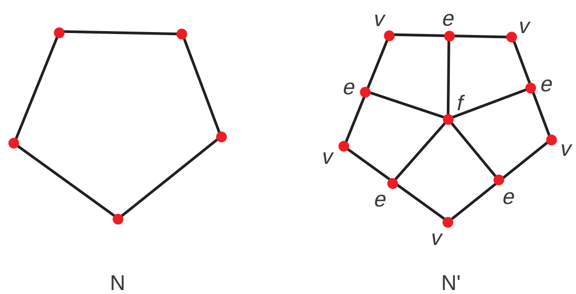

Figure 3.10. Primal-type refinement

A regular vertex in this setting is a vertex with valency 4, and a regular face is also of valency 4, namely, a quadrilateral face. Vertices or faces with valency $\neq 4$ are termed *irregular* or 'extraordinary'. In a closed net, every new e-vertex has valency 4. Every new v-vertex inherits the valency of its parent vertex, and every new f-vertex inherits the valency of its parent face. Therefore, the number of irregularities in a net remains constant throughout the subdivision process. Note that, after one subdivision iteration, all the faces are quadrilateral. The actual locations in $\mathbb{R}^3$ of the vertices V' are determined by the stencils of the subdivision scheme.

Catmull–Clark scheme. The first example of a primal-type scheme is the Catmull–Clark scheme (Catmull and Clark 1978, Doo and Sabin 1978), defined as an extension of the bicubic B-spline scheme (3.31) to closed nets of arbitrary topology. Its stencils are depicted in Figure 3.11.

The stencils for the new e-vertices and v-vertices involve the neighbouring new f-vertices (depicted as empty circles). The weight W_K in the stencil for the new v-vertex depends on the valency K of that vertex. Different formulae for W_K produce different limit surface behaviour near irregular vertices. A commonly used formula for W_K is

$$W_K = K(K-2), \quad K = 3, 4, \ldots .$$

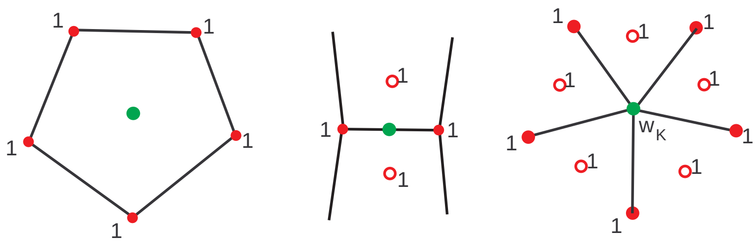

Figure 3.11. Catmull–Clark scheme:
f-stencil (left), e-stencil (middle) and v-stencil (right)

As long as $W_4 = 8$, the limit surfaces of this scheme are C^2 away from irregular points. Different variants of this scheme were investigated by Ball and Storry (1988, 1989). It is observed there that, for every choice of W_K, the surface curvature near an irregular point either tends to zero, or is unbounded. Applications of the Catmull–Clark scheme can be found in DeRose, Kass and Truong (1998) and Halstead, Kass and DeRose (1993).

Here we present an example (see Figure 3.12) of two surfaces generated from an initial triangulation, one by the Loop scheme and the other by the Catmull–Clark scheme, which regards the triangulation as a general net. Note that, in the latter case, most of the initial control points are irregular.

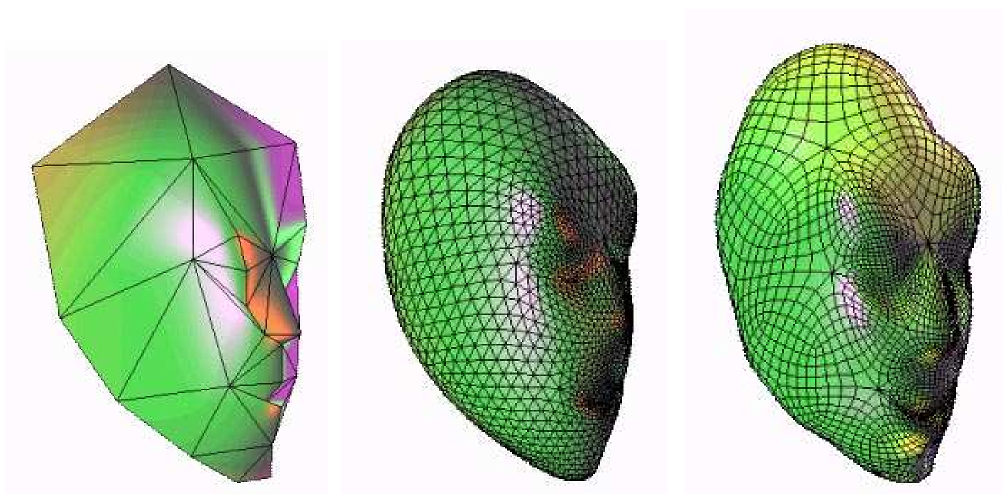

Figure 3.12. Head: initial control net (left),
two iterations with the Loop scheme (middle)
and with the Catmull–Clark scheme (right)

Dual-type refinement is depicted in Figure 3.13. Every new vertex in $v' \in V'$ corresponds to a pair $(v \in V, f \in F)$ such that v is a vertex of f in the original net N. It is considered a *dual* scheme, since vertices and edges in the original net $N = N(V, E, F)$ correspond to faces in the new net $N' = N(V', E', F')$. A regular vertex in this setting is a vertex with valency 4, and a regular face is a quadrilateral face.

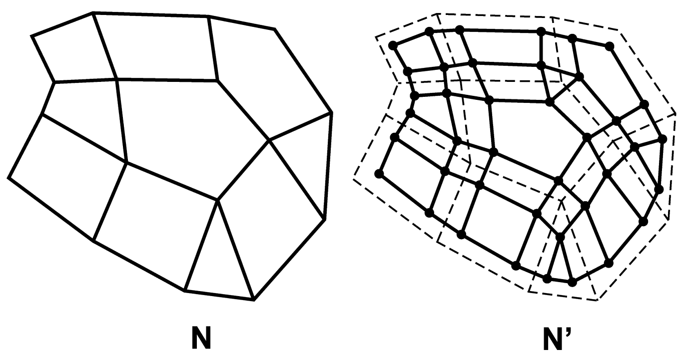

Figure 3.13. Dual-type refinement

Doo–Sabin scheme. The dual scheme due to Doo and Sabin generalizes the biquadratic B-spline scheme to subdivision of closed nets of arbitrary topological type.

The vertex v' is calculated by a weighted average of the vertices of f, with the stencils shown in Figure 3.14. The weights $\{S_j\}_{j=0,\dots,K-1}$ depend on the valency K of the face in the original net, and are given by

$$s_0 = \frac{K+5}{4K}, \qquad s_j = \frac{3 + 2\cos\left(\frac{2\pi j}{K}\right)}{4K}, \qquad j = 1, \dots, K-1.$$

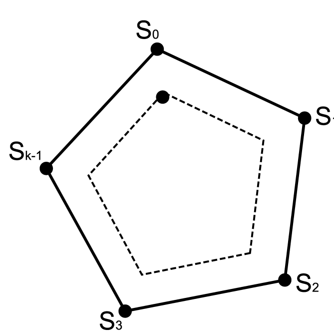

Figure 3.14. Doo–Sabin scheme: stencil for an f vertex

Almost everywhere the new nets are regular quadrilateral nets and the scheme reduces to the scheme defined by the symbol (3.30), giving the C^1 biquadratic spline surface.

In both examples, the Catmull–Clark scheme and the Doo–Sabin scheme, the mask parameters near an extraordinary vertex are chosen to achieve an overall C^1 limit surface. In Section 6 we describe the main results on the analysis of smoothness of stationary subdivision schemes near irregular vertices. Another dual-type subdivision scheme is 'the simplest scheme for smoothing polyhedra' presented in Peters and Reif (1997). In this scheme, given a polyhedron, a new polyhedron is constructed by connecting every edge-midpoint to its four neighbouring edge-midpoints. The limit surface is piecewise quadratic C^1 surface except at some extraordinary vertices. For additional material about subdivision schemes on general nets and their applications in computer graphics see Hoppe, DeRose, Duchamp, Halstead, Jin, McDonald, Schweitzer and Stuetzle (1994), Zorin *et al.* (1996), Zorin, Schröder and Sweldens (1997) and Zorin and Schröder (2000).

3.6. Further extensions

The inspiring iterative refinement idea, which is the basic concept in subdivision and in wavelets, has motivated many new research directions. In this section we briefly mention several extensions and generalizations of the uniform binary subdivision that are not discussed in this review. These include extensions to:

- non-uniform subdivision
- quasi-uniform and combined subdivision
- Lie group valued subdivision
- set-valued subdivision
- polyscale subdivision
- variational subdivision
- quasi-linear subdivision.

Non-uniform schemes. In many applications the data may be given on an irregular mesh and a scheme for iterative refinement of such data should be different from the standard uniform subdivision schemes. Also, convergence and smoothness analysis cannot be performed using the standard tools such as the z-transform or the Fourier transform. The tools that are being used for subdivision schemes over irregular grids are generalizations of the local matrix analysis (Section 5) and of the divided difference schemes (Section 4.2). See, for instance, Warren (1995*a*), Guskov (1998) and Daubechies, Guskov and Sweldens (1999). Another type of non-uniform scheme is still on uniform grids, but the subdivision refinement rules may differ from one point to the other. Here again it seems that the divided difference tools are the only way to analyse convergence and smoothness, as is done by Gregory and Qu (1996) for general corner cutting schemes. A systematic method for deriving the difference schemes, using a variation of the z-transform method, is presented in Levin (1999*e*). A general analysis of shape-preserving schemes for non-uniform data is done in Kuijt and van Damme (2002).

Quasi-uniform and combined subdivision. The analysis presented in this review is restricted to the case of closed nets, that is, there are no boundary edges. In real applications, there are boundaries of surface patches and boundaries may occur inside a patch if the patch should pass through a curve or a system of curves. For a subdivision scheme, a boundary treatment requires the definition of special rules in the vicinity of the boundary, and consequently, a special smoothness analysis. A subdivision scheme, together with special boundary rules, is called a *combined subdivision scheme* in A. Levin (1999*b*, 1999*c*). In these works, analysis tools for combined subdivision schemes are developed, and combined schemes, based on some of the most 'popular' bivariate schemes, are designed. The problem of matching boundary conditions or curve interpolation by subdivision surfaces is also treated in Nasri (1997*a*, 1997*b*) and A. Levin (1999*d*). A boundary may also be the border between two regions, or two patches, where in each patch a different uniform subdivision scheme is applied. This is termed *quasi-uniform* or *piecewise uniform*, and here also a special smoothness analysis is required, as presented in Dyn, Gregory and Levin (1995) for the univariate case and in A. Levin (1999*a*, 1999*c*) and Zorin, Biermann and A. Levin (2000) for surfaces.

Lie group valued subdivision. In some applications the data must lie on a manifold W in $\mathbb{R}^d$, and the limit function is also expected to be a function from $\mathbb{R}^s$ into W. The usual subdivision schemes are defined via linear averaging refinement rules that do not necessarily give points in W. In a recent work (Donoho and Stodden 2001), the general case of Lie group valued data is considered. The main approach is based on the fact that each Lie group has its associated Lie algebra, related through the exponential map, and the subdivision operations are performed in the Lie algebra and mapped back to the group by the exponential map.

Set-valued subdivision. For these schemes the initial data and the refined data generated by the scheme are sequences of sets in $\mathbb{R}^d$, and the limit function is a set-valued function. This is motivated by the problem of the reconstruction of 3D objects from their 2D cross-sections. The given data form a sequence of 2D cross-sections and the set-valued function describes a

3D object. Subdivision schemes for set-valued data require the definition of operations on sets and the study of notions of convergence and smoothness of set-valued functions. These issues, for convex sets using Minkowski averages, and for general compact sets using the 'metric average', are studied in Dyn and Farkhi (2000, 2001*a*, 2001*b*).

Polyscale subdivision. A subdivision scheme is a two-scale process, using data at one refinement level to compute the values at the next refinement level. In Dekel and Dyn (2001), poly-scale subdivision schemes are introduced. Such schemes compute the next refinement level from several previous levels, using several masks. This new idea is also related to the notion of poly-scale refinable functions, and opens up new theoretical convergence and smoothness issues. These issues, several interesting examples, and the relation of poly-scale subdivision schemes to matrix subdivision schemes, are presented in Dekel and Dyn (2001).

Variational subdivision. A variational approach to interpolatory subdivision is presented in Kobbelt (1996*b*). The resulting schemes are global, that is, every new point depends on all the points of the control polygon to be refined. The refinement is defined by minimizing a quadratic 'energy' functional, resulting in a 'fair' limit surface.

Quasi-linear subdivision. Quasi-linear schemes are nonlinear binary interpolatory schemes defined on a regular grid, with linear insertion rules which are data-dependent. In Cohen, Dyn and Matei (2001) a specific class based upon the weighted-ENO interpolation technique is analysed.

4. Convergence and smoothness analysis on regular grids

In this section, analysis of the (uniform) convergence of subdivision schemes on regular grids is presented, together with analysis of the smoothness of the limit functions.

First we present a method which relates the convergence and smoothness of nonstationary schemes to the convergence and smoothness of related stationary schemes (Dyn and Levin 1995); then we present a method for the analysis of stationary schemes, based on difference schemes (see Dyn (1992) and references therein). This method is also applied directly to certain nonstationary schemes.

The other main approaches to the convergence and smoothness analysis are in terms of Fourier transforms, and in terms of the joint spectral radius of a finite set of finite-dimensional matrices. The latter approach is briefly reviewed in Section 5.2. The Fourier analysis approach is not surveyed here: interested readers may consult Cohen and Conze (1992), Deslauriers and Dubuc (1989), Daubechies (1992) and Daubechies and Lagarias (1992*a*).

4.1. Analysis of nonstationary schemes via relations to stationary schemes

The analysis of the convergence of nonstationary schemes presented here, relies on the representation of a subdivision scheme $S_{\{\mathbf{a}^k\}}$ in terms of a sequence of operators $\{R_k : k \in \mathbb{Z}_+\}$ as in (2.22), where each R_k is defined by (2.21).

The main results are based on several properties of sequences of bounded linear operators in a Banach space. From now on all operators considered are bounded and linear. A sequence of operators $\{A_k : k \in \mathbb{Z}_+\}$ in a Banach space $\{X, \|\cdot\|\}$ defines the iterated process $x_{k+1} = A_k x_k,\ k \in \mathbb{Z}_+$, with $x_0 \in X$. Such a sequence is termed *convergent* if, for any $m \in \mathbb{Z}_+$ and any $x \in X$, $\lim_{k\to\infty} x_{m,k}$ exists, where $x_{m,k} = A_{m+k}\cdots A_{m+1}A_m x$. The sequence $\{A_k\}$ is termed *stable* if

$$\|A_{m+k}\cdots A_{m+1}A_m\| \le M < \infty, \quad \forall m, k \in \mathbb{Z}_+. \tag{4.1}$$

Two sequences of bounded operators $\{A_k\}$ and $\{B_k\}$ are called *asymptotically equivalent* if there exists $L \in \mathbb{Z}$, such that

$$\sum_{k=\max\{0,-L\}}^{\infty} \|A_{k+L} - B_k\| < \infty. \tag{4.2}$$

Proposition 4.1. Let $\{A_k\}$ and $\{B_k\}$ be asymptotically equivalent. Then $\{A_k\}$ is stable if and only if $\{B_k\}$ is stable.

The proof of this proposition (Dyn and Levin 1995) introduces the $\{A_k\}$-norms

$$\|x\|_m = \sup_k \|A_{m+k}\cdots A_m x\|, \quad m \in \mathbb{Z}_+,$$

which are equivalent to the norm of the Banach space when $\{A_k\}$ is stable. It also introduces the Banach spaces $X_m = \{X, \|\cdot\|_m\}$. The key observation is that A_m, as an operator from X_m to X_{m+1}, is bounded in norm by 1. From this observation follows Proposition 4.1. By similar reasoning we get the following.

Proposition 4.2. Let $\{A_k\}$ and $\{B_k\}$ be asymptotically equivalent. Then $\{A_k\}$ is stable and convergent if and only if so is $\{B_k\}$.

This analysis of sequences of operators in a Banach space leads to the important notion of 'asymptotic equivalence' between two subdivision schemes. Here we use the representation of subdivision schemes as operators on $X = C(\mathbb{R}^s)$, with the maximum norm. Two schemes $S_{\{\mathbf{a}^k\}}, S_{\{\mathbf{b}^k\}}$ are defined to be 'asymptotically equivalent' if, for some fixed $L \in \mathbb{Z}$,

$$\sum_{k=\max\{0,-L\}}^{\infty} \|\mathbf{a}^{k+L} - \mathbf{b}^k\|_\infty < \infty, \tag{4.3}$$

where $\|\mathbf{a}^k - \mathbf{b}^j\|_\infty = \max_{\alpha\in E^s} \sum_{\beta\in\mathbb{Z}^s} |a^k_{\alpha-2\beta} - b^j_{\alpha-2\beta}|$.

A scheme $S_{\{\mathbf{a}^k\}}$ is termed *stable* if there exists $M > 0$ such that, for all $k, j \in \mathbb{Z}_+$,

$$\|R_{j+k} \cdots R_{j+1} R_j\|_\infty < M, \tag{4.4}$$

with $\{R_j\}$ the operators corresponding to $S_{\{\mathbf{a}^k\}}$ as in (2.21). It is easy to conclude from (2.10) that a convergent scheme $S_{\{\mathbf{a}^k\}}$ is stable, if and only if the functions $\Phi_k = \sum_{\alpha\in\mathbb{Z}^s} |\phi_k(\cdot - \alpha)|$ are uniformly bounded for $k \in \mathbb{Z}_+$.

Two stable asymptotically equivalent schemes have similar convergence properties. This is easily concluded from Proposition 4.2.

Theorem 4.3. Let $S_{\{a^k\}}$ and $S_{\{b^k\}}$ be asymptotically equivalent. Then $S_{\{a^k\}}$ is stable and convergent if and only if $S_{\{b^k\}}$ is stable and convergent.

If $S_{\{\mathbf{b}^k\}} = S_{\mathbf{b}}$ is stationary, namely $\mathbf{b}^k = \mathbf{b}$ for $k \in \mathbb{Z}_+$, and $S_{\mathbf{b}}$ is convergent, then by (2.8) $S_{\mathbf{b}}$ is stable. Thus we have the following.

Corollary 4.4. Let $S_{\{\mathbf{a}^k\}}$ and $S_{\mathbf{b}}$ be asymptotically equivalent. If $S_{\mathbf{b}}$ is convergent then $S_{\{\mathbf{a}^k\}}$ is stable and convergent.

Example 1. As an example of convergence implied by Corollary 4.4, we consider the nonstationary subdivision scheme given by the symbols

$$a^k(z) = 2\prod_{i=1}^{m} \frac{1}{2}\left(1 + e^{\eta_i 2^{-k}} z\right), \quad k \in \mathbb{Z}_+, \tag{4.5}$$

with $\eta_1, \ldots, \eta_m$ distinct complex constants.

It is easy to verify that $S_{\{\mathbf{a}^k\}}$ is asymptotically equivalent to $S_{\mathbf{b}}$, whose symbol is

$$b(z) = 2^{-m+1}(1+z)^m. \tag{4.6}$$

Thus $S_{\mathbf{b}}$ is a convergent stationary subdivision scheme with basic limit function the polynomial B-spline of order m (degree $m-1$) with integer knots and support $[0, m]$ (see Section 3.1).

Thus the nonstationary scheme (4.5) is convergent. In fact its basic limit function is the exponential B-spline in $\mathrm{span}\{e^{\frac{\eta_i}{2}x} : 1 \le i \le m\}$ with integer knots and support $[0, m]$. (For more about exponential B-splines, see, for example, Schumaker (1980).)

One way to analyse the smoothness of the basic limit function of a nonstationary scheme $S_{\{\mathbf{a}^k\}}$ (and therefore all limit functions generated by $S_{\{a^k\}}$, as implied by (2.8)), is in terms of smoothing factors (Dyn and Levin 1995).

Theorem 4.5. Let the symbols of $S_{\{\mathbf{a}^k\}}$ be of the form

$$a^k(z) = \frac{1}{2}(1 + r_k z^\lambda) b^k(z), \quad k \ge K \in \mathbb{Z}_+, \tag{4.7}$$

with $\lambda \in \mathbb{Z}^s$, where $S_{\{\mathbf{b}^k\}}$ is a stable and convergent subdivision scheme with $\phi_{\{\mathbf{b}^k\}}$ of compact support and in $C^m(\mathbb{R}^s)$. If

$$r_k = e^{\eta 2^{-k}}(1+\varepsilon_k), \qquad \sum_{k=K}^{\infty} |\varepsilon_k| 2^k < \infty, \tag{4.8}$$

then $\phi_{\{\mathbf{a}^k\}}$ and $\partial_\lambda \phi_{\{\mathbf{a}^k\}}$ are in $C^m(\mathbb{R}^s)$

The factors $\frac{1}{2}(1+r_k z^\lambda)$ in (4.7) are termed *smoothing factors* and, for $\varepsilon_k = 0$ in (4.8), are related to the univariate elementary nonstationary scheme of (3.12),

Sketch of proof. The key to the proof is convolution property (2), which in this case has the form

$$\phi_{\{\mathbf{a}^k\}} = \int_{\mathbb{R}} \phi_{\{\mathbf{b}^k\}}(\cdot - \lambda t)\phi_{\{1+r_k z\}}(t)\,\mathrm{d}t.$$

Since $\phi_{\{1+r_k z\}}$ is supported on $[0,1]$ and is integrable (as discussed in Section 3.1), $\phi_{\{\mathbf{a}^k\}} \in C^m$. The result $\partial_\lambda \phi_{\{a^k\}} \in C^m$ follows from the general observation that, for a univariate integrable function h with $\sigma(h) = [0,1]$, and for a bounded continuous function $g \in C(\mathbb{R})$,

$$(g * h)(x) = \int_{x-1}^{x} g(t)h(x-t)\,\mathrm{d}t \in C^1(\mathbb{R}).$$

For a multivariate function, the conditions $\partial_\lambda f \in C^m$ for $\lambda \in \Lambda$, where Λ is a basis for $\mathbb{R}^s$, imply that $f \in C^{m+1}(\mathbb{R}^s)$. Hence Theorem 4.5 and convolution property (2) give us the following result.

Corollary 4.6. Let

$$a^k(z) = \prod_{i=1}^{s} \frac{1}{2}(1 + r_{i,k} z^{\lambda_i}) b^k(z), \quad k \geq K \in \mathbb{Z}_+,$$

where $S_{\{\mathbf{b}^k\}}$ satisfies the conditions of Theorem 4.5.

If, for $i = 1, \ldots, m$,

$$r_{i,k} = e^{\eta_i 2^{-k}}(1+\varepsilon_{i,k}), \qquad \sum_{k=K}^{\infty} |\varepsilon_{i,k}| 2^k < \infty,$$

and if $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}^s$ are linearly independent, then $\phi_{\{a^k\}} \in C^{m+1}$.

A good example where smoothness is deduced via Theorem 4.5 is provided by the nonstationary, univariate interpolatory schemes that reproduce finite-dimensional spaces of exponential polynomials (Dyn, Levin and Luzzatto 2001*a*).

Example 2. Consider finite-dimensional spaces of univariate exponential polynomials of the form

$$V_{\boldsymbol{\gamma},\boldsymbol{\mu}} = \operatorname{span}\{x^j e^{\gamma_\ell t}, j = 0, \dots, \mu_{\ell-1},\ \ell = 1, \dots, \nu\},$$

where $\boldsymbol{\gamma} = \{\gamma_1, \dots, \gamma_\nu\}$ are the roots with multiplicities $\boldsymbol{\mu} = \{\mu_1, \dots, \mu_\nu\}$ of a real polynomial of degree $N = \sum_{i=1}^{\nu} \mu_i$.

A scheme $S_{\{\mathbf{a}^k\}}$ is termed a *reproducing scheme* of $V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$, if, for any $k \in \mathbb{Z}_+$ and $\mathbf{f}^k = \{f_j^k = f(2^{-k}j) : j \in \mathbb{Z}\}$, with $f \in V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$,

$$S_{\mathbf{a}^k}\mathbf{f}^k = \mathbf{f}^{k+1}.$$

It is proved in Dyn *et al.* (2001*a*) that an interpolatory scheme $S_{\{a^k\}}$ with supports $\sigma(a^k)$ fixed for $k \in \mathbb{Z}_+$, which reproduces $V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$ and does not reproduce any bigger space of exponential polynomials containing $V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$, has the property that its symbols $\{a^k(z) : k \in \mathbb{Z}_+\}$ are Laurent polynomials of degree $2(N-1)$ satisfying

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r} a^k(z_n^k) = 2\delta_{0,r}, \quad \frac{\mathrm{d}^r}{\mathrm{d}z^r} a^k(-z_n^k) = 0,$$
$$r = 0, 1, \dots, \mu_n - 1, \quad n = 1, \dots, \nu, \tag{4.9}$$

where $z_n^k = \exp(2^{-(k+1)}\gamma_n)$, $n = 1, \dots, \nu$, $k \in \mathbb{Z}_+$.

For the case $N = 2n$, it can be concluded from (4.9) that the masks $\{\mathbf{a}^k : k \in \mathbb{Z}_+\}$ with $\sigma(\mathbf{a}^k) = [-n, n]$, tend as $k \to \infty$ to the mask $\mathbf{a}$ with $\sigma(\mathbf{a}) = [-n, n]$ of the interpolatory scheme, introduced in Deslauriers and Dubuc (1989), which reproduces the space π_n of all polynomials of degree not exceeding n (see Section 3.2). More specifically,

$$\|\mathbf{a}^k - \mathbf{a}\|_\infty < 2^{-k}B, \quad 0 < B < \infty,$$

and $a(z)$ is divisible by $(1+z)^n$, as follows from (4.9). Thus $S_{\{\mathbf{a}^k\}}$ is asymptotically equivalent to $S_{\mathbf{a}}$, and since $S_{\mathbf{a}}$ is convergent (Deslauriers and Dubuc 1989) so is $S_{\{\mathbf{a}^k\}}$. To conclude the smoothness of $\phi_0 = \phi_{\{\mathbf{a}^k\}}$ from the smoothness of $\phi_{\mathbf{a}}$, Theorem 4.5 is invoked. Assume $\phi_{\mathbf{a}} \in C^m$. Then, by the theory of smoothness of stationary schemes (see Section 4.2), $m \le n$. Consider, for each $k \in \mathbb{Z}_+$, the m linear factors of $a^k(z)$, $\prod_{i=1}^{m}(1 + (z_{n_i}^k)^{-1}z)$, where $n_1, \dots n_m$ are fixed integers in $\{1, \dots, \nu\}$, such that $\#\{n_i : n_i = j\} \le \mu_j$. The existence of these factors is guaranteed by (4.9). Each of the m factors divided by 2 is a smoothing factor. Now, the symbols $\{c^k(z) : k \in \mathbb{Z}_+\}$, given by

$$c^k(z) = \frac{a^k(z)2^m}{\prod_{i=1}^{m}(1 + (z_{n_i}^k)^{-1}z)}, \quad k \in \mathbb{Z}_+,$$

define a scheme $S_{\{\mathbf{c}^k\}}$ which is asymptotically equivalent to the scheme $S_{\mathbf{c}}$ with symbol

$$c(z) = \frac{a(z)2^m}{(1+z)^m}.$$

Since $\phi_{\mathbf{a}} \in C^m$, it follows from the analysis of stationary schemes (see Section 4.2) that $S_{\mathbf{c}}$ is convergent. Thus $S_{\{\mathbf{c}^k\}}$ is convergent, and by Theorem 4.5 and Corollary 4.6, $\phi_{\{\mathbf{a}^k\}} \in C^m$.

A specific example of this type is the following interpolatory 4-point scheme generating circles. In this scheme the insertion rule is constructed by interpolation with a function from the span of the four functions $H = \operatorname{span}\{1, t, \cos t, \sin t\}$.

The insertion rule turns out to be

$$\begin{aligned} f_{2j+1}^{k+1} &= \frac{-1}{16\cos^2(\theta 2^{-k-2})\cos(\theta 2^{-k-1})}(f_{j-1}^k + f_{j+2}^k) \\ &\quad + \frac{(1+2\cos(2\theta 2^{-k}))^2}{16\cos^2(\theta 2^{-k-2})\cos(\theta 2^{-k-1})}(f_j^k + f_{j+1}^k). \end{aligned}$$

Note that this insertion rule tends to the 4-point Dubuc–Deslauriers insertion rule as k tends to infinity, at the rate $O(2^{-k})$.

The above insertion rule together with $f_{2j}^{k+1} = f_j^k$, when applied to the equidistributed points on the circle, $\{f_j^0 = R(\cos(j\theta), \sin(j\theta))\}_{j=1}^N$, with $\theta = 2\pi/N$, generates denser sets of points on the circle.

Next we consider a similar example, but in the multivariate setting with general smoothing factors.

Example 3. Let

$$a^k(z) = 2^{s-\ell} \prod_{j=1}^{\ell} \left(1 + r_k^{(j)} z^{\lambda^{(j)}}\right), \quad k \in \mathbb{Z}_+,$$

be symbols with directions $\Lambda = \{\lambda^{(1)}, \ldots, \lambda^{(\ell)}\} \subset \mathbb{Z}^s$. If $r_k^{(1)}, \ldots, r_k^{(\ell)}$ satisfy (4.8), if the set Λ contains a subset of s directions with determinant ± 1, and if any subset of $\ell - m - 1$ directions spans $\mathbb{R}^s$, then $\phi_{\{\mathbf{a}^k\}}$ is in C^m.

To see this, observe that, under the conditions of the example, $S_{\{\mathbf{a}^k\}}$ is asymptotically equivalent to $S_{\mathbf{a}}$ with

$$a(z) = 2^s \prod_{\lambda^{(j)} \in \Lambda} (1 + z^{\lambda^{(j)}})/2.$$

By the conditions on Λ, $S_{\mathbf{a}}$ is convergent and $\phi_{\mathbf{a}}$ is the polynomial box-spline with directions Λ, which is C^m (see Section 3.1). Let $\Lambda_0 \subset \Lambda$ be the smallest subset of Λ for which $S_{\mathbf{b}}$ with $b(z) = 2^s \prod_{\lambda^{(j)} \in \Lambda_0} (1 + z^{\lambda^{(j)}})/2$ is C^0.

The scheme $S_{\{\mathbf{b}^k\}}$ with $b^k(z) = 2^s \prod_{\lambda^{(j)} \in \Lambda_0} (1 + r_k^{(j)} z^{\lambda^{(j)}})/2$ is asymptotically equivalent to $S_{\mathbf{b}}$. Hence, by Corollary 4.4, it follows that $\phi_{\{\mathbf{b}^k\}} \in C(\mathbb{R})$. The maximal m for which $S_{\mathbf{a}}$ is C^m is determined by repeated convolutions with respect to appropriate directions in $\Lambda \setminus \Lambda_0$. The same procedure of adding directions, in view of Theorem 4.5, proves that $S_{\{\mathbf{a}^k\}}$ is also C^m.

4.2. Analysis of univariate schemes via difference schemes

The case $s = 1$ is the simpler to analyse, and the theory for the stationary case is almost complete. This theory provides a method of analysis based on necessary and sufficient conditions for convergence, and in the most interesting cases, also necessary and sufficient conditions for smoothness.

The method presented here is general in the sense that it also applies to nonstationary schemes with symbols that are all divisible by the elementary factor $(1+z)$ and its powers, as in the stationary case. Yet, in the stationary case this divisibility is necessary and sufficient, while in the nonstationary case it is only sufficient.

A necessary condition for convergence (for any $s \in \mathbb{Z}_+ \backslash 0$) (Cavaretta *et al.* 1991, Dyn 1992), which is the key to this analysis in the univariate case, is easily derived from the stationary refinement step

$$f_\alpha^{k+1} = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha - 2\beta} f_\beta^k, \quad \alpha \in \mathbb{Z}^s.$$

Considering large k such that $|f_\alpha^j - (S_{\mathbf{a}}^\infty f^0)(2^{-j}\alpha)| < \varepsilon$, $j = k, k+1$ for sufficiently small ε, and taking into account that $\sigma(\mathbf{a})$ is finite, so that $2^{-k}\beta$ in the above sum is close to $2^{-k-1}\alpha$, we conclude the following.

Theorem 4.7. If $S_{\mathbf{a}}$ is (uniformly) convergent, then

$$\sum_{\beta \in \mathbb{Z}^s} a_{\alpha + 2\beta} = 1, \quad \alpha \in E^s, \tag{4.10}$$

where E^s are the extreme points of $[0,1]^s$.

Analysis of stationary schemes

For a stationary scheme $S_{\mathbf{a}}$ we identify the insertion rule $R_{\mathbf{a}}$ with the scheme. In the univariate case ($s = 1$) conditions (4.10) imply that $a(-1) = 0$, $a(1) = 2$. Thus $a(z)$ is divisible by $(1 + z)$, the elementary univariate factor of (3.1). As will become clear hereafter, $(1+z)/2$ is the stationary univariate smoothing factor.

Let the mask $\mathbf{a}$ satisfy (4.10). Then, $a(z) = (1+z)b(z)$, with $S_{\mathbf{b}}$ a scheme related to $S_{\mathbf{a}}$ by

$$S_{\mathbf{b}} \Delta \mathbf{f} = \Delta(S_{\mathbf{a}} \mathbf{f}), \tag{4.11}$$

where $\Delta\mathbf{f} = \{(\Delta\mathbf{f})_j = f_j - f_{j-1} : j \in \mathbb{Z}\}$. The verification of (4.11) is easily done in terms of the z-transform representation of subdivision schemes (2.25). Since

$$L(\Delta\mathbf{f}; z) = \sum_{j\in\mathbb{Z}} (\Delta\mathbf{f})_j \ z^j = (1-z)L(\mathbf{f}; z),$$

it follows from (2.25) and from the factorization of $a(z)$ that

$$\begin{aligned} L(\Delta\mathbf{f}^{k+1}; z) &= (1-z)a(z)L(\mathbf{f}^k; z^2) \\ &= b(z)(1-z^2)L(\mathbf{f}^k; z) \\ &= b(z)L(\Delta\mathbf{f}^k; z^2), \end{aligned}$$

which proves (4.11).

From now on we consider only masks that satisfy (4.10). It is clear that, if $S_\mathbf{a}$ is convergent, then $\lim_{k\to\infty}\sup_{j\in\mathbb{Z}} |\Delta f_j^k| = 0$ with $\mathbf{f}^k = S_\mathbf{a}^k\mathbf{f}^0$, or $\Delta\mathbf{f}^k = S_\mathbf{b}^k\Delta\mathbf{f}^0$. Thus, if $S_\mathbf{a}$ is convergent, then $S_\mathbf{b}$ maps any initial data to zero; in brief, it is contractive. The converse also holds.

Theorem 4.8. Let $a(z) = (1+z)b(z)$. $S_\mathbf{a}$ is convergent if and only if $S_\mathbf{b}$ is contractive.

Proof. It remains to prove that if $S_\mathbf{b}$ is contractive then $S_\mathbf{a}$ is convergent. Consider the sequence $\{F_k(t)\}_{k\in\mathbb{Z}_+}$ defined by (2.6). To show convergence of $S_\mathbf{a}$ it is sufficient to show that $\{F_k(t)\}_{k\in\mathbb{Z}_+}$ is a Cauchy sequence with respect to the sup-norm. Now by definition, and by the observation that a piecewise linear function attains its extreme values at its breakpoints

$$\sup_{t\in\mathbb{R}} |F_{k+1}(t) - F_k(t)| = \max\left\{ |\sup_{i\in\mathbb{Z}} |f_{2i}^{k+1} - g_{2i}^{k+1}|, \sup_{i\in\mathbb{Z}} |f_{2i+1}^{k+1} - g_{2i+1}^{k+1}| \right\}, \tag{4.12}$$

where

$$g_{2i}^{k+1} = f_i^k \qquad \text{and} \qquad g_{2i+1}^{k+1} = \frac{1}{2}(f_i^k + f_{i+1}^k). \tag{4.13}$$

It is easy to verify that (4.13) is represented in terms of the z-transform, by

$$L(\mathbf{g}^{k+1}; z) = \frac{(1+z)^2}{2z} L(\mathbf{f}^k; z^2).$$

Thus

$$\begin{aligned} L(\mathbf{f}^{k+1}; z) - L(\mathbf{g}^{k+1}; z) &= \left(a(z) - \frac{(1+z)^2}{2z}\right) L(\mathbf{f}^k; z^2) \\ &= (1+z)\left(b(z) - \frac{1+z}{2z}\right) L(\mathbf{f}^k; z^2) \\ &= (1+z)d(z)L(\mathbf{f}^k; z^2) \end{aligned}$$

with $d(z) = b(z) - \frac{1+z}{2z}$. Since, by (4.10), $a(1) = 2$, $d(1) = b(1) - 1 = 0$ and hence $d(z) = (1-z)e(z)$. This leads finally to

$$L(\mathbf{f}^{k+1} - \mathbf{g}^{k+1}; z) = e(z)(1-z^2)L(\mathbf{f}^k; z^2) = e(z)L(\Delta\mathbf{f}^k; z^2). \qquad (4.14)$$

Recalling that, by (4.12), $\|F_{k+1} - F_k\|_\infty = \sup_{j\in\mathbb{Z}} |f_j^{k+1} - g_j^{k+1}| = \|\mathbf{f}^{k+1} - \mathbf{g}^{k+1}\|_\infty$, and that, by (4.14) and (4.11),

$$\mathbf{f}^{k+1} - \mathbf{g}^{k+1} = S_{\mathbf{e}}\Delta\mathbf{f}^k = S_{\mathbf{e}}S_{\mathbf{b}}^k\Delta\mathbf{f}^0,$$

we finally get

$$\|F_{k+1} - F_k\|_\infty = \|\mathbf{f}^{k+1} - \mathbf{g}^{k+1}\|_\infty \le \|S_{\mathbf{e}}\|_\infty \|S_{\mathbf{b}}^k\Delta\mathbf{f}^0\|_\infty. \qquad (4.15)$$

Now, if $S_{\mathbf{b}}$ is contractive, namely if $S_{\mathbf{b}}^k\mathbf{f}$ tends to zero for all $\mathbf{f}$, then there exists $M \in \mathbb{Z}_+\backslash 0$ such that $\|S_b^M\|_\infty = \mu < 1$. Thus (4.15) leads to

$$\|F_{k+1} - F_k\|_\infty = \|\mathbf{f}^{k+1} - \mathbf{g}^{k+1}\|_\infty \le \|S_{\mathbf{e}}\|_\infty \mu^{[\frac{k}{M}]} \max_{0\le j<M} \|\Delta\mathbf{f}^j\| \le C\eta^k, \quad (4.16)$$

where $\eta = (\mu)^{\frac{1}{M}} < 1$ and C is a generic constant. Thus $\{F_k : k \in \mathbb{Z}_+\}$ is uniformly convergent. $\square$

With the analysis presented, we can design an algorithm for checking the convergence of $S_{\mathbf{a}}$ given the mask $\mathbf{a}$. Consider the iterated scheme $S_{\mathbf{b}}^\ell$, transforming data at level k to data at level $\ell + k$. Recall that the symbol of $S_{\mathbf{b}}^\ell$ can be computed by (2.28) as $b^{[\ell]}(z) = \prod_{j=1}^{\ell} b(z^{2^{j-1}})$, and thus, to check the contractivity of $S_{\mathbf{b}}$, the norms of $S_{\mathbf{b}}^\ell$, $\ell = 1, 2, \ldots$, have to be evaluated in terms of $b^{[\ell]}(z) = \sum_{j\in\mathbb{Z}} b_j^{[\ell]} z^k$, according to

$$\|S_b^\ell\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}} |b_{i-2^\ell j}^{[\ell]}| : 0 \le i < 2^\ell\right\}. \qquad (4.17)$$

The norm in (4.17) reflects the fact that there are 2^ℓ different rules in the iterated scheme $S_{\mathbf{b}}^\ell$:

$$\mathbf{g}^{k+\ell} = S_{\mathbf{b}}^\ell \mathbf{g}^k \Leftrightarrow g_i^{k+\ell} = \sum_{j\in\mathbb{Z}} b_{i-2^\ell j}^{[\ell]} g_j^k, \quad i \in \mathbb{Z}.$$

Schemes for which $S_{\mathbf{b}}$ is contractive, but $\|S_{\mathbf{b}}^\ell\|_\infty \ge 1$ for large ℓ ($\ell > 5$), are of no practical value, since a large number of iterations is required to observe convergence (small $\|\Delta\mathbf{f}^k\|_\infty$). Thus the algorithm has an input parameter M_0, such that if $\|S_{\mathbf{b}}^\ell\|_\infty \ge 1$ for $1 \le \ell \le M_0$, the scheme is declared to be practically 'not convergent'. A reasonable choice of M_0 is in the range $5 < M_0 \le 10$.

Algorithm for verifying convergence given the symbol.

Let $a(z)$ be the symbol of the scheme.

If $a(-1) \neq 0$, or $a(1) \neq 2$, then the scheme does not converge. Stop!

Compute $b^{[1]}(z) = a(z)/(1+z) = \sum_j b_j^{[1]} z^j$.

For $\ell = 1, \ldots, M_0$ $\{$

Compute $N_\ell = \max_{0 \le i < 2^\ell} \sum_{j \in \mathbb{Z}} |b^{[\ell]}_{i-2^\ell j}|$.

If $N_\ell < 1$, the scheme is convergent. Stop!

If $N_\ell \ge 1$, compute $b^{[\ell+1]}(z) = b^{[1]}(z) b^{[\ell]}(z^2) = \sum_{j \in \mathbb{Z}} b_j^{[\ell+1]} z^j$.

$\}$ End loop.

$S_{\mathbf{b}}$ is not contractive after M_0 iterations. Stop!

The parameters μ, M from the proof of Theorem 4.8 corresponding to a mask $\mathbf{a}$, also determine the Hölder exponent of $\phi_{\mathbf{a}}$ (or any $S_{\mathbf{a}}^\infty f^0$), and the rate of convergence of the subdivision scheme.

Theorem 4.9. Let $\mathbf{a}, \mu, M, \eta$, be as in the proof of Theorem 4.8, and define $\nu = -(\log_2 \mu)/M$. Then

$$|\phi_{\mathbf{a}}(y) - \phi_{\mathbf{a}}(x)| \le C|x-y|^\nu.$$

Moreover, the rate of convergence of the sequence $\{F_k(t)\}_{k \in \mathbb{Z}_+}$ defined in (2.6) is

$$\|F_k(t) - S_{\mathbf{a}}^\infty \mathbf{f}^0\|_\infty \le C\eta^k.$$

Here C is a generic constant.

Proof. Both claims of the theorem follow from (4.16). The second follows directly with the aid of the observation

$$|(S_{\mathbf{a}}^\infty \mathbf{f}^0 - F_k)(x)| = \lim_{\ell \to \infty} |F_\ell(x) - F_k(x)| \le \sum_{j=k}^\infty |F_{j+1}(x) - F_j(x)|.$$

To verify the first claim, we use the second claim in the bound

$$|\phi_{\mathbf{a}}(x) - \phi_{\mathbf{a}}(y)| \le |\phi_{\mathbf{a}}(x) - F_k(x)| + |\phi_a(y) - F_k(y)| + |F_k(x) - F_k(y)|,$$

and the obvious bound

$$|F_k(x) - F_k(y)| \le 2\|\Delta \mathbf{f}^k\|_\infty,$$

both holding for any k. The first claim now follows by estimating $\Delta f^k = S_{\mathbf{b}}^k \Delta \boldsymbol{\delta}$ in terms of $\|S_{\mathbf{b}}^M\|_\infty < \mu$, and by the observation that, for $2^{-k} \le |x-y| \le 2^{-k+1}$,

$$\mu^{[\frac{k}{M}]} \le C\mu^{\frac{k}{M}} = C2^{-k\nu} \le C|x-y|^\nu. \qquad \square$$

The tools for the analysis of smoothness are similar to the tools for the convergence analysis. The analysis of smoothness is based on the observation that in the stationary case $(1+z)/2$ is a smoothing factor.

Theorem 4.10. Let $a(z) = \frac{1+z}{2} q(z)$. If $S_{\mathbf{q}}$ is convergent and C^ℓ, then $S_{\mathbf{a}}$ is convergent and $C^{\ell+1}$.

Sketch of proof. By convolution property (2) and by (3.2), $S_{\mathbf{a}}$ is convergent, and

$$\phi_{\mathbf{a}}(x) = \int_{x-1}^{x} \phi_{\mathbf{q}}(t)\,\mathrm{d}t. \tag{4.18}$$

Thus

$$\phi'_{\mathbf{a}}(x) = \phi_{\mathbf{q}}(x) - \phi_{\mathbf{q}}(x-1). \tag{4.19}$$

Theorem 4.10 supplies a sufficient condition for smoothness. A repeated use of Theorem 4.10 together with Theorem 4.8 leads to

Corollary 4.11. Let $a(z) = \frac{(1+z)^{m+1}}{2^m} b(z)$ with $S_{\mathbf{b}}$ contractive. Then $\phi_{\mathbf{a}} \in C^m(\mathbb{R})$. Moreover

$$\phi_{\mathbf{a}}^{(\ell)} = S^{\infty}_{a(z)(1+z)^{-\ell}2^{\ell}} \Delta^\ell \boldsymbol{\delta}, \quad \ell = 0, 1, \ldots, m,$$

where $\Delta^\ell = \Delta\Delta^{\ell-1}$ is defined recursively.

For a scheme with symbol $a(z) = 2^{-m}(1+z)^{m+1}b(z)$, instead of finding the maximal ℓ such that $S_{a(z)2^{\ell-1}(1+z)^{-\ell}}$ is contractive, Rioul (1992) suggests computing the numbers $\|S^\ell_{\mathbf{b}}\|_\infty = \mu_\ell$ and $\nu_\ell = -(\log_2 \mu_\ell)/\ell$. If $m - \nu_\ell > 0$, then $\phi_{\mathbf{a}} \in C^{[m-\nu_\ell]}$. Defining $\nu = \sup_{\ell \geq 1} \nu_\ell$, if $m - \nu > 0$, then $\phi_{\mathbf{a}} \in C^{[m-\nu]}$, and $\phi_{\mathbf{a}}^{([m-\nu])}$ has Hölder exponent $n - \epsilon$ for any $\epsilon > 0$ with $n = m - \nu - [m-\nu]$.

Example 4. Consider the stationary interpolatory 4-point scheme with symbol (3.19)

$$a_w(z) = \frac{1}{2z}(1+z)^2[1 - 2wz^{-2}(1-z)^2(z^2+1)].$$

By Theorem 4.8, the range of w for which $S_{\mathbf{a}_w}$ is convergent is the range for which $S_{\mathbf{b}_w}$, with symbol

$$b_w(z) = \frac{1}{2z}(1+z)[1 - 2wz^{-2}(1-z)^2(z^2+1)],$$

is contractive. The condition $\|S_{\mathbf{b}_w}\|_\infty < 1$ yields the range

$$-\frac{3}{8} < w < \frac{-1+\sqrt{13}}{8},$$

while the condition $\|S^2_{\mathbf{b}_w}\|_\infty < 1$ yields the range

$$-\frac{1}{4} < w < \frac{-1+\sqrt{17}}{8}.$$

Thus a range of w for which $S_{\mathbf{a}_w}$ is convergent is (Dyn *et al.* 1991)

$$-\frac{3}{8} < w < \frac{-1+\sqrt{17}}{8} \cong 0.39.$$

By Corollary 4.11, it is sufficient to show that $S_{\mathbf{c}_w}$, with symbol

$$c_w(z) = \frac{1}{z}[1 - 2wz^{-2}(z-1)^2(z^2+1)],$$

is contractive, in order to prove that $S_{\mathbf{a}_w}$ is C^1. Now, $\|R_{\mathbf{c}_w}\|_\infty \geq 1$, while $\|R^2_{\mathbf{c}_w}\|_\infty < 1$ for $0 < w < \frac{\sqrt{5}-1}{8}$, as is shown in Dyn *et al.* (1991).

The fact that $\phi_{\mathbf{a}_w} \notin C^2(\mathbb{R})$ can be deduced from necessary conditions that are violated (see Section 5.2). In Daubechies and Lagarias (1992*a*), it is shown, by methods as in Section 5.2, that $\phi'_{\mathbf{a}_w}$ is differentiable except at all the dyadic points in its support.

After deriving similar results to the above for a class of nonstationary schemes, we return to the stationary case, and show that, in most interesting cases, if $\phi_{\mathbf{a}} \in C^m(\mathbb{R})$ then necessarily the symbol $a(z)$ is divisible by $(1+z)^{m+1}$. In this sense the form of $a(z)$ in Corollary 4.11 is necessary for $S_{\mathbf{a}}$ with C^m limit functions. This result holds if $\phi_{\mathbf{a}}$ is L_∞-stable, namely if for any bounded bi-infinite sequence $\mathbf{f} = \{f_i : i \in \mathbb{Z}\}$,

$$C_2 \sup_{i\in\mathbb{Z}} |f_i| \leq \left\| \sum_{i\in\mathbb{Z}} f_i \phi(x-i) \right\|_\infty \leq C_1 \sup_{i\in\mathbb{Z}} |f_i|, \tag{4.20}$$

with $0 < C_2 \leq C_1 < \infty$. For most interesting schemes the basic limit function is L_∞-stable, for example, for interpolatory schemes and for spline schemes. We also study the related property that, for $S_{\mathbf{a}}$ with $\phi_{\mathbf{a}} \in C^m$ and L_∞-stable, π_m is invariant under $_{\mathbf{a}}$.

Analysis of nonstationary schemes with symbols divisible by stationary smoothing factors

In this section the tools of analysis of Section 4.2 are extended to a class of nonstationary schemes. Theorem 4.10 also holds for a nonstationary scheme with symbols

$$a^k(z) = \frac{(1+z)}{2} q^k(z), \quad k \in \mathbb{Z}_+, \quad k \geq K,$$

with K some positive integer, and such that $S_{\{\mathbf{q}^k\}}$ is convergent. A version of Theorem 4.8 also holds in the nonstationary case. It supplies only a sufficient condition for convergence.

Theorem 4.12. Let a nonstationary scheme be given by the symbols

$$a^k(z) = (1+z)b^k(z), \quad k \in \mathbb{Z}_+, \quad k \geq K \in \mathbb{Z}_+.$$

If $S_{\{\mathbf{b}^k\}}$ is contractive then $S_{\{\mathbf{a}^k\}}$ is convergent.

This theorem holds since $R_{\mathbf{a}^k}$ and $R_{\mathbf{b}^k}$, defined by (2.2) and (2.3), are related by

$$\Delta R_{\mathbf{a}^k}\mathbf{f} = R_{\mathbf{b}^k}\Delta\mathbf{f}, \quad k \in \mathbb{Z}_+, \quad k \geq K, \tag{4.21}$$

and therefore, by the same arguments as in the stationary case, the contractivity of $S_{\{\mathbf{b}^k\}}$ implies the convergence of $S_{\{\mathbf{a}^k\}}$. A simple sufficient condition for the contractivity of $S_{\{\mathbf{b}^k\}}$ is

$$\|R_{\mathbf{b}^k}\|_\infty = \max\left(\sum_{j\in\mathbb{Z}} |b^k_{i-2j}| : i \in \{0,1\}\right) \leq \mu < 1, \quad k \in \mathbb{Z}_+, \quad k \geq K, \tag{4.22}$$

since then, for $\mathbf{g}^k = R_{\mathbf{b}^{k-1}}R_{\mathbf{b}^{k-2}}\cdots R_{\mathbf{b}^0}\mathbf{g}^0$, we have $\|\mathbf{g}^k\|_\infty \leq \mu^k\|\mathbf{g}^0\|_\infty$.

From Theorem 4.12 and the remark above it, we conclude the following.

Corollary 4.13. Let a nonstationary scheme be given by the symbols

$$a^k(z) = \frac{(1+z)^{m+1}}{2^m}b^k(z), \quad k \in \mathbb{Z}_+, \quad k \geq K \in \mathbb{Z}_+.$$

If $S_{\{\mathbf{b}^k\}}$ is contractive then $S_{\{\mathbf{a}^k\}}$ is C^m.

Example 5. In this example we study properties of the up-function introduced in Section 3.1, by applying the analysis tools of this section.

Let a nonstationary scheme be given by the symbols, as in (3.5),

$$a^k(z) = \frac{(1+z)^k}{2^{k-1}}, \quad k \in \mathbb{Z}_+.$$

To show that $\phi_{\{\mathbf{a}^k\}} \in C^\infty(\mathbb{R})$, we show that $\phi_{\{\mathbf{a}^k\}} \in C^m(\mathbb{R})$, for any $m \in \mathbb{Z}_+$.

Now, for $k \geq m+2$,

$$a^k(z) = \frac{(1+z)^{m+1}}{2^m} \cdot \frac{(1+z)^{k-m-1}}{2^{k-m-1}},$$

and, by Corollary 4.13, $\phi_{\{\mathbf{a}^k\}} \in C^m$ if $S_{\{\mathbf{b}^k\}}$ is contractive, with

$$b^k(z) = \frac{(1+z)^{k-m-1}}{2^{k-m-1}}, \quad k \in \mathbb{Z}_+, \quad k \geq m+2.$$

But $\|R_{\mathbf{b}^k}\|_\infty = \frac{1}{2}$ for $k \in \mathbb{Z}_+$, $k \geq m+2$, which proves that $S_{\{\mathbf{b}^k\}}$ is contractive.

Next we show that $\sigma(\phi_{\{\mathbf{a}^k\}}) = [0,2]$. Using (2.11) we get, from (3.5),

$$\sigma(\phi_{\{\mathbf{a}^k\}}) = \sum_{j=0}^{\infty} 2^{-j-1}\sigma(\mathbf{a}^j) = \sum_{j=0}^{\infty} 2^{-j-1}[0, j+1] = [0,2].$$

Polynomials generated by stationary schemes
For stationary interpolatory schemes in $\mathbb{R}^s$ it is easy to show (Dyn and Levin 1990) that $\phi_{\mathbf{a}} \in C^m$ implies that π_m is reproduced by the scheme, namely

$$R_{\mathbf{a}} p|_{\mathbb{Z}^s} = p\left(\frac{\cdot}{2}\right)|_{\mathbb{Z}^s}, \quad \text{and} \quad S_{\mathbf{a}}^{\infty} p|_{\mathbb{Z}^s} = p, \quad \text{for } p \in \pi_m(\mathbb{R}^s). \tag{4.23}$$

For a subdivision scheme with a stable basic limit function, the proof is more involved. It was first proved in Cavaretta *et al.* (1991). Here we present a proof for $s = 1$, which is extendable to univariate matrix subdivison schemes (Dyn and Levin 2002) and to multivariate schemes.

The proof is based on the following important observation in Warren (1995*a*).

Theorem 4.14. Let $S_{\mathbf{a}}$ be a C^m-convergent univariate, stationary subdivision scheme. Let $\mathbb{B}$ denote the set of bi-infinite sequences, and let $\mathbf{v} = \{v_j : j \in \mathbb{Z}\} \in \mathbb{B}$ be an eigenvector of $R_{\mathbf{a}}$ with eigenvalue λ, that is,

$$R_{\mathbf{a}}\mathbf{v} = \lambda \mathbf{v}. \tag{4.24}$$

Then the following hold.

(1) If $|\lambda| \geq 2^{-m}$, either $S_{\mathbf{a}}^{\infty}\mathbf{v} \equiv 0$ or $S_{\mathbf{a}}^{\infty}\mathbf{v} = x^i$ for some $0 \leq i \leq m$, and $\lambda = 2^{-i}$. Also $\lambda = 2^{-i}$, $0 \leq i \leq m$, cannot have a generalized eigenvector $\mathbf{u} \in \mathbb{B}$, satisfying

$$R_{\mathbf{a}}\mathbf{u} = \lambda\mathbf{u} + \mathbf{v}. \tag{4.25}$$

(2) If $|\lambda| < 2^{-m}$ then $(S_{\mathbf{a}}^{\infty}\mathbf{v})^{(\ell)}(0) = 0$, $\ell = 0, \dots, m$.

(3) If $\lambda \neq 2^{-i}$, $0 \leq i \leq m$, and $\mathbf{u}$ is a corresponding generalized eigenvector satisfying (4.25), then $(S_{\mathbf{a}}^{\infty}\mathbf{u})^{(\ell)}(0) = 0$, $\ell = 0, \dots, m$.

The proof of Theorem 4.14 is based on the relations

$$(S_{\mathbf{a}}^{\infty}\mathbf{v})(x) = \lambda(S_{\mathbf{a}}^{\infty}\mathbf{v})(2x), \quad (S_{\mathbf{a}}^{\infty}\mathbf{u})(x) = \lambda(S_{\mathbf{a}}^{\infty}\mathbf{u})(2x) + (S_{\mathbf{a}}^{\infty}\mathbf{v})(2x)$$

for $\mathbf{v}, \mathbf{u}$ satisfying (4.24) and (4.25) respectively, and on the continuity at $x = 0$ of the derivatives of order up to m of $S_{\mathbf{a}}^{\infty}\mathbf{u}$, $S_{\mathbf{a}}^{\infty}\mathbf{v}$.

A direct consequence of Theorem 4.14 deals with polynomials generated by a univariate stationary subdivision scheme with smooth limit functions (Dyn *et al.* 1995).

Theorem 4.15. Let $S_{\mathbf{a}}$ be a C^m-subdivision scheme. Then there exist $\mathbf{v}^{[i]} \in \mathbb{B}$, $i = 0, \ldots, m$, such that

$$R_{\mathbf{a}}\mathbf{v}^{[i]} = 2^i \mathbf{v}^{[i]}, \quad S_{\mathbf{a}}^{\infty}\mathbf{v}^{[i]} = x^i, \quad i = 0, \ldots, m. \tag{4.26}$$

The argument leading to (4.26) is that 2^{-i} must be an eigenvalue of $R_{\mathbf{a}}$ for $i = 0, \ldots, m$, otherwise there exists $\ell \in \{0, 1, \ldots, m\}$ such that $2^{-\ell}$ is not an eigenvalue of $R_{\mathbf{a}}$, implying that $\phi_{\mathbf{a}}^{(\ell)} \equiv 0$, in view of Theorem 4.14. But $\phi_{\mathbf{a}}$ is of compact support, $\phi_a \not\equiv 0$, which contradicts $\phi_{\mathbf{a}}^{(\ell)} \equiv 0$. Next we show that $\mathbf{v}^{[i]}$ in Theorem 4.15 is of the form $\mathbf{v}^{[i]} = x^i|_{\mathbb{Z}} + p_i|_{\mathbb{Z}}$ with $p_i \in \pi_{i-1}$, $i = 0, \ldots, m$ (here $p_0 \equiv 0$). For this proof the L_∞-stability of $\phi_{\mathbf{a}}$ is needed. We term a scheme L_∞-stable if its basic limit function is L_∞-stable.

Theorem 4.16. Let $S_{\mathbf{a}}$ be C^m and L_∞-stable. Then there exist polynomials $p_i \in \pi_{i-1}$, $i = 0, \ldots, m$, with $p_0 \equiv 0$, such that

$$S_{\mathbf{a}}^{\infty}(x^i + p_i)|_{\mathbb{Z}} = x^i, \quad i = 0, \ldots, m. \tag{4.27}$$

Sketch of proof. The case $i = 0$ follows directly from (4.10), because $R_{\mathbf{a}}$ maps the constant sequence $\mathbf{1} = \mathbf{u} = \{u_j = 1 : j \in \mathbb{Z}\}$ on itself.

In the following we indicate the proof for $i = 1$. For $i = 2, \ldots, m$, the proof is similar. Let $\mathbf{v} = \mathbf{v}^{[1]}$ satisfy $S_{\mathbf{a}}^{\infty}\mathbf{v} = x$, and for $r \in \mathbb{Z}_+ \backslash 0$ let $\Delta^{(r)}\mathbf{v} = \{v_{j+r} - v_j : j \in \mathbb{Z}\}$. Then the linearity and uniformity of $S_{\mathbf{a}}$ leads to $S_{\mathbf{a}}^{\infty}\Delta^{(1)}\mathbf{v} = x + 1 - x = 1$ or

$$S_{\mathbf{a}}^{\infty}(\Delta^{(1)}\mathbf{v} - \mathbf{1}) \equiv 0. \tag{4.28}$$

If $\Delta^{(1)}\mathbf{v} - \mathbf{1} \in \mathbb{B}$ is bounded, then by the L_∞-stability of $\phi_{\mathbf{a}}$, $\Delta^{(1)}\mathbf{v} = \mathbf{1}$, which is equivalent to $\mathbf{v} = x|_{\mathbb{Z}} + c\mathbf{1}$ for some $c \in \mathbb{R}$. Thus the claim of the theorem for $i = 1$ follows. To show the boundedness of $\Delta^{(1)}\mathbf{v} - \mathbf{1}$ we consider (4.28) at the integers, which in view of (2.8) has the form

$$\sum_{j \in \mathbb{Z}} \big((\Delta^{(1)}\mathbf{v})_j - 1\big)\phi_{\mathbf{a}}(n - j) = 0, \quad n \in \mathbb{Z}. \tag{4.29}$$

Equation (4.29) can be regarded as a finite difference equation for $\Delta^{(1)}\mathbf{v} - \mathbf{1}$, since $\phi_{\mathbf{a}}|_{\mathbb{Z}}$ is finitely supported, and is not identically equal to zero (otherwise $\phi_{\mathbf{a}} \equiv 0$ by (2.15)). As a solution of (4.29), $\Delta^{(1)}\mathbf{v} - \mathbf{1}$ is either bounded or it grows at least polynomially as $j \to \infty$ or $j \to -\infty$. For the latter possibility, $\mathbf{v}$ would have faster than linear growth. This possibility is eliminated, since

$$(R_{\mathbf{a}}\Delta^{(r)}\mathbf{v})_\alpha = \sum_{j \in \mathbb{Z}} a_{\alpha - 2j}(v_{j+r} - v_r) = \frac{1}{2}v_{\alpha+2r} - \frac{1}{2}v_\alpha = \frac{1}{2}(\Delta^{(2r)}\mathbf{v})_\alpha,$$

from which it is concluded, in view of (4.28), that

$$S_{\mathbf{a}}^{\infty}\Delta^{(1)}\mathbf{v} = \lim_{\ell \to \infty} 2^{-\ell}\Delta^{(2^\ell)}\mathbf{v} = 1,$$

or that $v_{\pm 2^\ell} = v_0 \pm 2^\ell + o(1)$, which is in contradiction to faster than linear growth.

As a direct consequence of Theorem 4.16 we get the following result.

Corollary 4.17. Let $S_{\mathbf{a}}$ be C^m and L_∞-stable. Then $\pi_m|_{\mathbb{Z}}$ is invariant under $R_{\mathbf{a}}$ and, for $p \in \pi_i$, $0 \leq i \leq m$,

$$R_{\mathbf{a}} p|_{\mathbb{Z}} = q\left(\frac{\cdot}{2}\right)|_{\mathbb{Z}},$$

with $q \in \pi_i$ and $p - q \in \pi_{i-1}$, while $p = q$ for $i = 0$.

In the following subsection we derive the factorization of the symbol of a scheme satisfying the requirements of Corollary 4.17.

Factorization of symbols of stationary, smooth, L_∞-stable schemes, and related necessary conditions

First we show that, if $S_{\mathbf{a}}$ is C^m and L_∞-stable, then its symbol has the factor $(1+z)^{m+1}$. Later we show that, necessarily, $S_{2^m a(z)(1+z)^{-m-1}}$ is contractive. A similar result holds for L_p-stability and convergence in the L_p-norm, $1 \leq p \leq \infty$ (Jia 1995). These results are important in the analysis of smoothness of univariate stationary schemes (see Section 4.2).

Theorem 4.18. Let $S_{\mathbf{a}}$ be C^m and L_∞-stable. Then

$$a(z) = (1+z)^{m+1} b(z) \tag{4.30}$$

with $b(z)$ a Laurent polynomial.

Proof. We use a recursive construction of 'divided difference' schemes with symbols

$$a^{[i]}(z) = 2^i (z+1)^{-i} a(z), \quad i = 1, \ldots, m+1.$$

If $a^{[i]}(z)$ is a Laurent polynomial, then, in view of (4.11), $S_{\mathbf{a}^{[i]}}$ is related to $S_{\mathbf{a}}$ by

$$S_{\mathbf{a}^{[i]}} d_k^i \mathbf{f} = d_{k+1}^i S_{\mathbf{a}} \mathbf{f}, \quad \mathbf{f} \in \mathbb{B},$$

where $d_k^i \mathbf{f} = (2^k)^i \Delta^i f$ is the sequence of divided differences of order i on refinement level k. Since, by Corollary 4.17, $R_{\mathbf{a}}$ maps $\mathbf{1} \in \mathbb{B}$ to itself, $\sum_{i \in \mathbb{Z}} a_{2i} = \sum_{i \in \mathbb{Z}} a_{2i+1} = 1$ and $a(z)$ is divisible by $(1+z)$. This guarantees that $a^{[1]}$ exists. Now, $R_{\mathbf{a}}$ maps $\mathbf{v} = x|_{\mathbb{Z}}$, to $R_{\mathbf{a}} \mathbf{v} = \frac{1}{2} x|_{\mathbb{Z}} + c\mathbf{1}$ for some $c \in \mathbb{R}$, so $R_{\mathbf{a}^{[1]}}$ maps $\mathbf{1} \in \mathbb{B}$ into itself. Thus $a^{[1]}(z)$ is divisible by $(1+z)$, and $a^{[2]}$ exists. The general argument is similar.

By applying $(2^k)^i \Delta^i \mathbf{f}$ to $\mathbf{f} = x^i|_{\mathbb{Z}}$ we get a constant sequence. This sequence is mapped by $R_{a^{[i]}}$ to $(2^{k+1})^i \Delta^i R_{\mathbf{a}} \mathbf{f}$, which is the same constant sequence. This is the case since $R_{\mathbf{a}}\mathbf{f} = (\frac{\cdot}{2})^i + q(\frac{\cdot}{2})$ with $q \in \pi_{i-1}$ by Corollary 4.17, and $\Delta^i q(\frac{\cdot}{2}) = 0$. Again, if $R_{a^{[i]}}$ maps the constant sequence on itself, then $a^{[i]}(z)$ is divisible by $(1+z)$.

Using this argument for $i = 0, 1, \ldots, m$ we conclude that $a^{[i]}$ exists for $i = 1, \ldots, m+1$, and thus (4.30) holds. $\square$

Example 6. Consider the symbol

$$a(z) = \frac{1}{4}(1+z)(1+z^2)^2 = \frac{1}{4}(1+z+2z^2+2z^3+z^4+z^5). \tag{4.31}$$

It is easy to see that (4.10) holds, since $a(1) = 2$, $a(-1) = 0$. To verify that $S_{\mathbf{a}}$ is convergent, we show that $S_{\mathbf{b}}$ with $b(z) = \frac{1}{4}(1+z^2)^2$ is contractive. Now, $b(z) = \frac{1}{4}(1+2z^2+z^4)$ and therefore $\|S_{\mathbf{b}}\|_\infty = 1$. Yet from (2.28) and (4.17) we get

$$\begin{aligned} b^{[2]}(z) &= \frac{1}{16}(1+z^4)^2(1+z^2)^2 \\ &= \frac{1}{16}(1+2z^2+3z^4+4z^6+3z^8+2z^{10}+z^{12}), \end{aligned}$$

and therefore $\|S^2_{\mathbf{b}}\|_\infty = \frac{1}{2}$.

Since the symbol $c(z) = 1+z^2$ satisfies $c(1) = 2$, $S_{\mathbf{c}}$ converges weakly (Derfel *et al.* 1995). It is easy to verify that $S^\infty_{\mathbf{c}}\boldsymbol{\delta} = \frac{1}{2}\chi_{[0,1]}$ in the sense of weak convergence. By convolution property (3)

$$\phi_{\mathbf{a}} = \tfrac{1}{4}\chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]}.$$

Thus $\phi_{\mathbf{a}} \in C^1$, while $a(z)$ is not divisible by $(1+z)^2$. This indicates, in view of Theorem 4.18, that $\phi_{\mathbf{a}}$ is not L_∞-stable. Indeed, consider the sequence $\mathbf{u} = \{u_i = (-1)^i : i \in \mathbb{Z}\}$. Clearly $\mathbf{u}$ is bounded. Now in view of (4.31), $R_{\mathbf{a}}\mathbf{u} = \mathbf{0} \in \mathbb{B}$, and therefore $S^\infty_{\mathbf{a}}\mathbf{u} = \sum_{i\in\mathbb{Z}}(-1)^i\phi_{\mathbf{a}}(\cdot - i) \equiv 0$, and $\phi_{\mathbf{a}}$ is not L_∞-stable.

Once we have the factorization of the symbol of a stationary C^m, L_∞-stable scheme,

$$a(z) = (1+z)^{m+1}b(z),$$

we can show that $\frac{2^m a(z)}{(1+z)^{m+1}}$ is the symbol of a contractive scheme. For that we need two results, which are of importance beyond their current use.

Theorem 4.19. Let ϕ be a solution of the functional equation

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}} a_\alpha \phi(2x-\alpha), \tag{4.32}$$

with a mask $\mathbf{a}$ satisfying (4.10). If ϕ is compactly supported, continuous and L_∞-stable, then $S_{\mathbf{a}}$ is convergent.

This theorem was first proved in Cavaretta *et al.* (1991). Here we give a sketch of a different proof (Dyn and Levin 2002).

Sketch of proof. Recalling the relation in (2.16), we observe that, since $\phi = T_{\mathbf{a}}\phi$, and $\mathbf{a} = R_{\mathbf{a}}\boldsymbol{\delta}$,

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} (R_{\mathbf{a}}\delta)_\alpha \phi(2x-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} (R^k_{\mathbf{a}}\delta)_\alpha \phi(2^k x-\alpha), \tag{4.33}$$

and that for all $k \in \mathbb{Z}_+$

$$\sum_{\alpha\in\mathbb{Z}} \phi(x-\alpha) = \sum_{\alpha\in\mathbb{Z}} (R^k_{\mathbf{a}}\mathbf{1})_\alpha \phi(2^k x-\alpha) = \sum_{\alpha\in\mathbb{Z}} \phi(2^k x-\alpha). \tag{4.34}$$

The continuity and L_∞-stability of ϕ together with (4.34) leads, after proper normalization, to

$$\sum_{\alpha\in\mathbb{Z}} \phi(\cdot-\alpha) \equiv 1. \tag{4.35}$$

Combining (4.33) and (4.35) we get

$$0 = \sum_{\alpha\in\mathbb{Z}} \phi(2^k x-\alpha)\big[(R^k_{\mathbf{a}}\delta)_\alpha - \phi(x)\big],$$

which, together with the continuity, compact support and L_∞-stability of ϕ, yields

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}\cap K} |(R^k_{\mathbf{a}}\boldsymbol{\delta})_\alpha - \phi(2^{-k}\alpha)| = 0,$$

for any compact set $K \subset \mathbb{R}$. This is the convergence of $S_{\mathbf{a}}$ in the sense of (2.5) to a continuous limit function ϕ, hence uniform convergence.

The second theorem is taken from Dyn and Levin (2002), where it is proved for matrix masks.

Theorem 4.20. Let $a(z) = \frac{1+z}{2} q(z)$, with $S_{\mathbf{a}}$ L_∞-stable and C^1. Then

$$\varphi = \sum_{\alpha\in\mathbb{Z}} \phi'_{\mathbf{a}}(\cdot-\alpha)$$

is a continuous, L_∞-stable solution of

$$\varphi(x) = T_{\mathbf{q}}\varphi(x) = \sum_{\alpha\in\mathbb{Z}} q_\alpha \varphi(2x-\alpha). \tag{4.36}$$

Sketch of proof. The function φ is well defined, continuous and of compact support. It is related to $\phi_{\mathbf{a}}$ by

$$\phi_{\mathbf{a}}(x) = \int_{x-1}^{x} \varphi(t)\,\mathrm{d}t = \varphi * \chi_{[0,1]}. \tag{4.37}$$

Suppose φ is not L_∞-stable; then there exists a bounded nonzero sequence $\mathbf{u} \in \mathbb{B}$ such that

$$\sum_{\alpha\in\mathbb{Z}} u_\alpha \varphi(\cdot - \alpha) \equiv 0.$$

By integrating this relation from $x-1$ to x we obtain

$$\sum_{\alpha\in\mathbb{Z}} \mathbf{u}_\alpha \phi_{\mathbf{a}}(x - \alpha) \equiv 0.$$

This last relation contradicts the L_∞-stability of $\phi_{\mathbf{a}}$. Thus φ is also L_∞-stable. To verify that $\varphi = T_{\mathbf{q}}\varphi$, we observe that $\phi_{\mathbf{a}} = T_{\mathbf{a}}\phi_{\mathbf{a}}$, and after taking the Fourier transform, it is equivalent to

$$\hat{\phi}_{\mathbf{a}}(w) = \frac{1}{2}\hat{a}\left(\frac{w}{2}\right)\hat{\phi}_{\mathbf{a}}\left(\frac{w}{2}\right) \tag{4.38}$$

with $\hat{a}(w) = \sum_{\alpha\in\mathbb{Z}} a_\alpha e^{-iw\alpha}$. Now by (4.37) $\hat{\varphi}(w)\frac{1-e^{-iw}}{w} = \hat{\phi}_{\mathbf{a}}(w)$. Multiplying (4.38) by $\frac{w}{1-e^{-iw}}$, we obtain

$$\hat{\varphi}(w) = \frac{1}{2}\,\frac{2\hat{a}(\frac{w}{2})}{1+e^{-i\frac{w}{2}}}\hat{\varphi}\left(\frac{w}{2}\right) = \frac{1}{2}\hat{q}\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right),$$

proving (4.36).

From Theorems 4.19, 4.20, 4.18 and 4.8 we conclude the following.

Corollary 4.21. Let $S_{\mathbf{a}}$ be C^1 and L_∞-stable. Then $b(z) = \frac{2a(z)}{(1+z)^2}$ is a Laurent polynomial and $S_{\mathbf{b}}$ is contractive.

This corollary together with Corollary 4.11 implies the following.

Corollary 4.22. Let $S_{\mathbf{a}}$ be convergent and L_∞-stable. Then the contractivity of $S_{2^m a(z)(1+z)^{-(m+1)}}$ is necessary and sufficient for $S_{\mathbf{a}}$ to be C^m.

4.3. Analysis of bivariate stationary schemes via difference schemes

The analysis of convergence and smoothness of multivariate subdivision schemes defined on regular grids, which is of interest to geometric modelling in $\mathbb{R}^3$, is in the case $s = 2$. Thus, for the sake of simplicity of presentation, we limit the discussion to this case. The results are easily extended to $s > 2$. Here we present similar analysis tools to those in the univariate, stationary case for bivariate, stationary subdivision schemes defined on regular quad-meshes and on regular triangulations. When the symbol factorizes into sufficiently many linear factors (each a univariate smoothing factor in some direction in $\mathbb{Z}^2$), the analysis is almost as simple as in the univariate case (Cavaretta *et al.* 1991, Dyn 1992). This factorization is not the result of (4.10) or of the smoothness of the limit functions, as in the univariate case, but is an additional assumption, which holds for many of the schemes in use.

In fact the same factorization of nonstationary symbols leads to similar results, even for nonstationary schemes. When the symbol is not factorizable to univariate smoothing factors, (4.10) leads to non-unique matrix difference schemes, and the theory of the univariate case can be extended to this case (Cavaretta *et al.* 1991, Dyn 1992, Hed 1990); see Section 4.3.

Analysis of schemes with factorizable symbols

The necessary conditions for convergence of a bivariate scheme $S_{\mathbf{a}}$ defined on $\mathbb{Z}^2$, which are obtained from (4.10), are

$$\sum_{\beta\in\mathbb{Z}^2} a_{\alpha-2\beta} = 1, \quad \alpha \in \big\{(0,0),(0,1),(1,0),(1,1)\big\}. \tag{4.39}$$

These conditions imply

$$a(1,1) = 4, \qquad a(-1,1) = 0, \qquad a(1,-1) = 0, \qquad a(-1,-1) = 0. \tag{4.40}$$

In contrast to the univariate case ($s = 1$), in the bivariate case ($s = 2$), the necessary conditions (4.39) and the derived conditions on $a(z)$, (4.40), do not imply a factorization of the mask to linear factors.

If the factorization

$$a(z) = (1+z_1)^m(1+z_2)^m b(z), \quad z = (z_1,z_2), \tag{4.41}$$

is imposed, then, with $m = 1$, the convergence can be analysed almost as in the univariate case, and similarly the smoothness if $m > 1$.

Theorem 4.23. Let $S_{\mathbf{a}}$ have a symbol of the form (4.41) with $m = 1$. If the schemes with the symbols $a_1(z) = \frac{a(z)}{1+z_1} = (1+z_2)b(z)$, $a_2(z) = \frac{a(z)}{1+z_2} = (1+z_1)b(z)$ are both contractive, then $S_{\mathbf{a}}$ is convergent. Conversely, if $S_{\mathbf{a}}$ is convergent then $S_{\mathbf{a}_1}$ and $S_{\mathbf{a}_2}$ are contractive.

The proof of this theorem is similar to the proof of Theorem 4.8, due to the observation that for $\Delta_1\mathbf{f} = \{f_{i,j} - f_{i-1,j} : i,j \in \mathbb{Z}\}$, and $\Delta_2\mathbf{f} = \{f_{i,j} - f_{i,j-1} : i,j \in \mathbb{Z}\}$,

$$S_{\mathbf{a}_\ell}\Delta_\ell\mathbf{f} = \Delta_\ell S_{\mathbf{a}}\mathbf{f}, \quad \ell = 1,2.$$

Thus convergence is checked in this case as contractivity of two subdivision schemes $S_{\mathbf{a_1}}, S_{\mathbf{a_2}}$. For schemes having the symmetry of the square grid (topologically equivalent rules for the computation of vertices corresponding to edges), then $a_1(z_1,z_2) = a_2(z_2,z_1)$, and the contractivity of only one scheme has to be checked. Note that the factorization in (4.41) has then the symmetry of $\mathbb{Z}^2$.

For the smoothness result, we introduce the inductive definition of differences: $\Delta^{[i,j]} = \Delta_1\Delta^{[i-1,j]}$, $\Delta^{[i,j]} = \Delta_2\Delta^{[i,j-1]}$, $\Delta^{[1,0]} = \Delta_1$, $\Delta^{[0,1]} = \Delta_2$.

Theorem 4.24. Let $a(z)$ be factorizable as in (4.41). If the schemes with the masks

$$a_{i,j}(z) = \frac{2^{i+j}a(z)}{(1+z_1)^i(1+z_2)^j}, \quad i,j = 0,\ldots,m \tag{4.42}$$

are convergent, then

$$\frac{\partial^{i+j}}{\partial t_1^i \partial t_2^j}(S_{\mathbf{a}}^\infty \mathbf{f}^0)(t) = (S_{\mathbf{a_{i,j}}}^\infty \Delta_1^i \Delta_2^j \mathbf{f}^0)(t), \quad i,j = 0,\ldots,m. \tag{4.43}$$

In particular, $S_{\mathbf{a}}$ is C^m.

In geometric modelling the required smoothness of surfaces is at least C^1 and at most C^2. To verify that a scheme $S_{\mathbf{a}}$ generates C^1 limit functions, with the aid of the last two theorems, we have to assume a symbol of the form

$$a(z) = (1+z_1)^2(1+z_2)^2 b(z),$$

and to check the contractivity of the three schemes with symbols

$$2(1+z_1)(1+z_2)b(z), \qquad 2(1+z_2)^2 b(z), \qquad 2(1+z_1)^2 b(z).$$

This analysis applies also to tensor product schemes, but is not needed, since if $a(z) = a_1(z_1)a_2(z_2)$ is the symbol of a tensor product scheme, then $\phi_{\mathbf{a}}(t_1,t_2) = \phi_{\mathbf{a_1}}(t_1)\cdot\phi_{\mathbf{a_2}}(t_2)$, and its smoothness properties are derived from those of $\phi_{\mathbf{a_1}}, \phi_{\mathbf{a_2}}$.

Similar results hold for schemes defined on regular triangulations. For the topology of a regular triangulation, we regard the subdivision scheme as operating on the 3-directional grid. (The vertices of $\mathbb{Z}^2$ with edges in the directions $(1,0),(0,1),(1,1)$.)

Since the 3-directional grid can be regarded also as $\mathbb{Z}^2$, (4.39) and (4.40) hold for convergent schemes on this grid.

A scheme for regular triangulations treats each edge in the 3-directional grid in the same way with respect to the topology of the grid. The symbol of such a scheme, when factorizable, has the form

$$a(z) = (1+z_1)^m(1+z_2)^m(1+z_1z_2)^m b(z). \tag{4.44}$$

Example 7. The symbol of the butterfly scheme on the 3-directional grid has the form (Dyn, Levin and Micchelli 1990*b*)

$$a(z) = \frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2)(1-wc(z_1,z_2))(z_1z_2)^{-1} \tag{4.45}$$

with

$$\begin{aligned} c(z_1,z_2) = {} & 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1} \\ & + 2z_1^{-1}z_2 + 2z_1z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2. \end{aligned} \tag{4.46}$$

Convergence analysis for schemes with factorizable symbols of the form (4.44) is similar to that for schemes with symbols of the form (4.41).

Theorem 4.25. Let $S_{\mathbf{a}}$ have the symbol

$$a(z) = (1 + z_1)(1 + z_2)(1 + z_1 z_2)b(z). \tag{4.47}$$

Then $S_{\mathbf{a}}$ is convergent if and only if the schemes with symbols

$$a_1(z) = \frac{a(z)}{1 + z_1}, \qquad a_2(z) = \frac{a(z)}{1 + z_2}, \qquad a_3(z) = \frac{a(z)}{1 + z_1 z_2} \tag{4.48}$$

are contractive. If any two of these schemes are contractive, then the third is also contractive.

Note that

$$S_{\mathbf{a_3}} \Delta_3 \mathbf{f} = \Delta_3 S_{\mathbf{a}} \mathbf{f},$$

with $(\Delta_3 \mathbf{f})_{i,j} = f_{i,j} - f_{i-1,j-1}$. Thus, if two of the schemes $S_{\mathbf{a_i}}$, $i = 1, 2, 3$ are contractive then the differences in two linearly independent directions tend to zero as $k \to \infty$, which implies, as in the proof of Theorem 4.8, the uniform convergence of the bilinear interpolants to $\{\mathbf{f}^k\}_{k \in \mathbb{Z}_+}$.

The smoothness analysis for a scheme with a symbol (4.47) is different from that for schemes with symbols as in (4.41).

Theorem 4.26. Let $S_{\mathbf{a}}$ have the symbol (4.47), and let $a_i(z)$, $i = 1, 2, 3$ be as in (4.48). Then $S_{\mathbf{a}}$ generates C^1 limit functions, if the schemes with the symbols $2a_i(z)$, $i = 1, 2, 3$, are convergent. If any two of these schemes are convergent then the third is also convergent. Moreover,

$$\frac{\partial}{\partial t_i}(S_{\mathbf{a}}^\infty \mathbf{f}^0)(t) = (S_{\mathbf{2a_i}} \Delta_i \mathbf{f}^0)(t), \quad i = 1, 2,$$
$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)(S_{\mathbf{a}}^\infty \mathbf{f}^0)(t) = (S_{\mathbf{2a_3}} \Delta_3 \mathbf{f}^0)(t).$$

The verification, based on Theorems 4.25 and 4.26, that the scheme $S_{\mathbf{a}}$ with symbol (4.47) is C^1, requires us to check the contractivity of the three schemes with symbols

$$2(1 + z_1)b(z), \qquad 2(1 + z_2)b(z), \qquad 2(1 + z_1 z_2)b(z).$$

If these three schemes are contractive, then $S_{\mathbf{a}}$ generates C^1-limit functions. For $a(z)$ with the symmetries of the 3-directional grid, it is sufficient to check the contractivity of only one of the three schemes, as is easily observed in the next example.

Example 8. To verify that the butterfly scheme generates C^1-limit functions, we use the fact that the symbol $a(z)$ of the butterfly scheme, given in (4.45), is of the form (4.47). In view of the observation following

Theorem 4.26, we have to check the contractivity of the three schemes with symbols

$$q_i(z) = (1+z_i)\big(1-wc(z_1,z_2)\big)(z_1z_2)^{-1}, \quad i=1,2,$$
$$q_3(z) = (1+z_1z_2)\big(1-wc(z_1,z_2)\big)(z_1z_2)^{-1}.$$

Noting that

$$c(z_1,z_2) = c(z_2,z_1) = c(z_1z_2, z_1^{-1}),$$

and that the factor $(z_1z_2)^{-1}$ in a symbol does not affect the norm of the corresponding subdivision operator, it is sufficient to verify the contractivity of $S_{\mathbf{r}}$, where

$$r(z) = (1+z_1)\big(1-wc(z_1,z_2)\big) = \sum_{\alpha\in\mathbb{Z}^2} r_\alpha z^\alpha.$$

Now

$$\|S_{\mathbf{r}}\|_\infty = \max_{\ell,k\in\{0,1\}}\left(\sum_{i,j\in\mathbb{Z}} |r_{k+2i,\ell+2j}|\right),$$

and since

$$\sum_{i,j\in\mathbb{Z}} |r_{2i,2j}| = |1-8w| + |8w|,$$

$\|S_r\|_\infty \geq 1$ for all values of w.

Next, we show that for sufficiently small $w>0$, $\|S_{\mathbf{r}}^2\|_\infty < 1$ (Dyn *et al.* 1990*b*). Ignoring coefficients of $r^{[2]}(z)$ that are not $O(1)$, and computing the others up to order $O(w)$, we get

$$\begin{aligned} r^{[2]}(z) &= r(z)r(z^2) \\ &= (1+z_1+z_1^2+z_1^3)\big(1-wc(z_1,z_2)-wc(z_1^2,z_2^2)+O(w^2)\big) \\ &= \sum_{i,j\in\mathbb{Z}} r_{ij}^{[2]} z_1^i z_2^j. \end{aligned}$$

Thus, for $j\neq 0$, $r_{i,j}^{[2]} = O(w)$ while $r_{i,0}^{[2]} = 1+O(w)$, $i=0,1,2,3$. From this we conclude that it is sufficient to show that, for sufficiently small w,

$$\sum_{i,j\in\mathbb{Z}} \big|r_{\ell+4i,4j}^{[2]}\big| < 1, \quad \ell=0,1,2,3.$$

When $\ell=0$, all the nonzero coefficients $\{r_{4i,4j}^{[2]}\}$ are

$$\begin{aligned} r_{0.0}^{[2]} &= 1-16w+O(w^2), \\ r_{4,0}^{[2]} &= 8w+O(w^2), \\ r_{4,4}^{[2]} &= r_{0,-4}^{[2]} = -2w+O(w^2). \end{aligned}$$

Hence, for sufficiently small $w > 0$,

$$\sum_{i,j\in\mathbb{Z}} \left|r^{[2]}_{4i,4j}\right| = |1-16w| + 12|w| + O(w^2) < 1.$$

When $\ell = 1$, the relevant coefficients are

$$\begin{aligned} r^{[2]}_{1,0} &= 1 - 12w + O(w^2), \\ r^{[2]}_{5,0} &= 4w + O(w^2), \\ r^{[2]}_{5,4} &= r^{[2]}_{1,-4} = -2w + O(w^2), \end{aligned}$$

and, for sufficiently small $w > 0$,

$$\sum_{i,j\in\mathbb{Z}} |r_{1+4i,4j}| = |1-12w| + 8|w| + O(w^2) < 1.$$

The cases $\ell = 2$ and $\ell = 3$ are similar to the cases $\ell = 1$ and $\ell = 0$, respectively. Thus, for sufficiently small $w > 0$, the limit surfaces/functions generated by the butterfly scheme on regular triangulations are C^1.

An explicit value of w_0, such that for $w \in (0, w_0)$ the butterfly scheme generates C^1 limit functions on regular triangulations, is computed in Gregory (1991). The computation shows that $w_0 > \frac{1}{16}$. The value $w = \frac{1}{16}$ is of special importance, since for this value the butterfly scheme on $\mathbb{Z}^2$ reproduces cubic polynomials, while for $w \neq \frac{1}{16}$ the scheme reproduces only linear polynomials. These properties are related to the approximation properties of the scheme (see Section 7).

Analysis of general schemes defined on $\mathbb{Z}^2$

The necessary conditions in the bivariate case (4.39) imply four conditions on the symbol (4.40).

These four conditions lead to a subdivision scheme with a matrix mask, for the vector of first differences

$$\mathbf{\Delta f} = \left\{ (\mathbf{\Delta f}) = \left(\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \mathbf{f} \right)_{ij} = \begin{pmatrix} f_{ij} - f_{i-1,j} \\ f_{ij} - f_{i,j-1} \end{pmatrix} : (i,j) \in \mathbb{Z}^2 \right\}. \tag{4.49}$$

Contrary to the univariate case, this matrix mask is *not uniquely* determined. The matrix mask can be derived with the help of the following lemma.

Lemma 4.27. Let $p(z) = p(z_1, z_2)$ be a Laurent polynomial satisfying

$$p(1,1) = p(-1,1) = p(1,-1) = p(-1,-1) = 0. \tag{4.50}$$

Then there exist Laurent polynomials, p_1, p_2, such that

$$p(z) = (1 - z_1^2)p_1(z) + (1 - z_2^2)p_2(z). \tag{4.51}$$

The 'factorization' in (4.51) is not unique, since the term $(1-z_1^2)(1-z_2^2)q(z)$, with q a Laurent polynomial, can be added to the first term on the right-hand side of (4.51) and subtracted from the second.

The proof of the lemma is based on the following two observations.

(a) The Laurent polynomial

$$P(z) = \frac{1}{2}\big[(1+z_2)p(z_1,1) + (1-z_2)p(z_1,-1)\big]$$

coincides with $p(z)$ for $z_2 = 1$ and $z_2 = -1$, and therefore there exists a Laurent polynomial $r(z)$ such that

$$p(z) - P(z) = (1-z_2^2)r(z).$$

(b) $P(z)$ is a Laurent polynomial that is divisible by $(1-z_1^2)$, since, in view of (4.50), $P(\pm 1, z_2) \equiv 0$.

The last lemma guarantees the 'factorization' assumed in (4.52).

Theorem 4.28. Let $a(z) = a(z_1, z_2)$ satisfy (4.40), and let

$$\begin{aligned}(1-z_1)a(z) &= b_{11}(z)(1-z_1^2) + b_{12}(z)(1-z_2^2),\\ (1-z_2)a(z) &= b_{11}(z)(1-z_1^2) + b_{22}(z)(1-z_2^2),\end{aligned} \tag{4.52}$$

where $b_{ij}, i, j = 1, 2$, are Laurent polynomials. Then

$$\mathbf{\Delta} R_{\mathbf{a}}\mathbf{f} = R_{\mathbf{B}}\mathbf{\Delta f}, \tag{4.53}$$

where $R_{\mathbf{B}}$ is the refinement rule

$$(R_{\mathbf{B}}\mathbf{v})_\alpha = \sum_{\beta\in\mathbb{Z}^2} B_{\alpha-2\beta}v_\beta, \quad \alpha \in \mathbb{Z}^2, \tag{4.54}$$

with the matrix symbol

$$B(z) = \sum_{\alpha\in\mathbb{Z}^2} B_\alpha z^\alpha = \begin{pmatrix} b_{11}(z) & b_{12}(z)\\ b_{21}(z) & b_{22}(z)\end{pmatrix}, \tag{4.55}$$

and with $\mathbf{v}$ a bi-infinite sequence of vectors in $\mathbb{R}^2$, that is,

$$\mathbf{v} = \{v_\alpha : v_\alpha \in \mathbb{R}^2, \alpha \in \mathbb{Z}^2\}.$$

Sketch of proof. The formalism of the z-transform is the tool for proving the theorem. Observing that

$$L(\mathbf{\Delta f}; z) = \begin{pmatrix} 1-z_1\\ 1-z_2\end{pmatrix} L(\mathbf{f}; z),$$

and recalling the basic relation in (2.25),

$$L(R_{\mathbf{a}}\mathbf{f}; z) = a(z)L(\mathbf{f}; z^2),$$

we obtain from (4.52)

$$\begin{pmatrix} 1-z_1 \\ 1-z_2 \end{pmatrix} L(R_{\mathbf{a}}\mathbf{f};z) = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} \begin{pmatrix} 1-z_1^2 \\ 1-z_2^2 \end{pmatrix} L(\mathbf{f};z^2),$$

which is equivalent to (4.53) and (4.54). In the following we let $S_{\mathbf{B}}$ denote the stationary scheme with the refinement rule $R_{\mathbf{B}}$ in (4.54). Theorem 4.28 leads, as in the univariate case, to the following result.

Corollary 4.29. Let $S_{\mathbf{a}}$ be a bivariate subdivision scheme satisfying (4.39). Then $S_{\mathbf{a}}$ is convergent if and only if $S_{\mathbf{B}}$ is contractive for all initial data of the form $\mathbf{\Delta f}$.

A sufficient condition for convergence is thus the contractivity of the scheme $S_{\mathbf{B}}$. This can be verified by considering the numbers $\|S_{\mathbf{B}}^M\|_\infty$ for $M = 1, 2, \ldots$. Here again the formalism of the z-transform leads to the symbol

$$B^{[M]}(z) = B(z)B^{[M-1]}(z^2) = B(z)B(z^2)\cdots B(z^{2^{M-1}})$$

of S_B^M, where the order of the factors in the matrix product is significant. The norm of $S_{\mathbf{B}}^M$ is given by (Hed 1990, Dyn 1992)

$$\|S_{\mathbf{B}}^M\|_\infty = \max_{\alpha\in E_2^M} \left\| \sum_{\beta\in\mathbb{Z}^2} |B^{[M]}_{\alpha-2^M\beta}| \right\|_\infty$$

where $|A|$ denotes the matrix whose elements are the absolute values of the corresponding elements in the matrix A, $\|A\|_\infty$ denotes the L_∞-norm of the matrix A, and where $E_2^M = \{\alpha = (\alpha_1, \alpha_2) : 0 \le \alpha_1 < 2^M, 0 \le \alpha_2 < 2^M\}$. Thus a similar algorithm to the one given in the univariate case (see Section 4.2), applies also in the bivariate case, although it is based only on a sufficient condition and on a non-unique 'factorization'. It is possible to use optimization techniques to find, among all possible 'factorizations', the one that minimizes $\min\{\|S_{\mathbf{B}}^M\|_\infty : 1 \le M \le 10\}$ (Kasas 1990).

The C^1 analysis is based on the following result.

Theorem 4.30. Let $S_{\mathbf{a}}$ be a convergent subdivision scheme. If $2S_{\mathbf{B}}$ with $\mathbf{B}$ given by (4.55) and (4.52) is convergent for initial data of the form $\mathbf{\Delta f}$, then $S_{\mathbf{a}}$ is C^1.

This result is analogous to Theorem 4.10 in the univariate case. Furthermore,

$$\big(2S_{\mathbf{B}}\big)^\infty \mathbf{\Delta f}^0 = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} S_{\mathbf{a}}^\infty \mathbf{f}^0. \tag{4.56}$$

Equation (4.56) only holds if

$$\sum_{\alpha\in\mathbb{Z}^2} B_{\gamma-2\alpha} = I_{2\times 2}, \quad \gamma \in \{(0,0),(0,1),(1,0),(1,1)\}, \tag{4.57}$$

which follows from the linear independence of the two components of

$$\big(2S_{\mathbf{B}}\big)^{\infty}\boldsymbol{\Delta}\mathbf{f} = (\partial_1 S_{\mathbf{a}}^{\infty}\mathbf{f}, \partial_2 S_{\mathbf{a}}^{\infty}\mathbf{f})^T$$

for generic $\mathbf{f}$. From (4.57) and Lemma 4.27 follows the existence of a matrix subdivision scheme $S_{\mathbf{C}}$, for the vectors $2^{-k}\Delta^2\mathbf{f}^k$,

$$\boldsymbol{\Delta}^2\mathbf{f} = \begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix}\boldsymbol{\Delta}\mathbf{f} \in \mathbb{R}^4,$$

with $\mathbf{C}$ a mask of matrices of order 4×4 with symbol

$$\begin{pmatrix}C^{(1,1)}(z) & C^{(1,2)}(z)\\ C^{(2,1)}(z) & C^{(2,2)}(z)\end{pmatrix},$$

where $C^{(i,j)}(z)$ is a matrix of order 2×2 defined by the 'factorization'

$$\begin{pmatrix}1-z_1\\ 1-z_2\end{pmatrix}2b_{ij}(z) = C^{(i,j)}(z)\begin{pmatrix}1-z_1^2\\ 1-z_2^2\end{pmatrix}.$$

If $S_{\mathbf{C}}$ is contractive then $S_{2\mathbf{B}}$ is convergent and $S_{\mathbf{a}}$ is C^1. The same ideas can be further extended to deal with higher orders of smoothness (Hed 1990, Dyn 1992).

5. Analysis by local matrix operators

Given masks $\{\mathbf{a}^k\}$ of the same finite support, the corresponding refinement rules (2.2) and their representations in matrix form (2.18) are local. For the subdivision scheme $S_{\{\mathbf{a}^k\}}$, this locality is also expressed by the compact supports of the corresponding basic limit functions $\{\phi_k : k \in \mathbb{Z}_+\}$, and the representations (2.10) of the limit functions $S_k^{\infty}\mathbf{f}^0$.

5.1. The local matrix operators in the univariate setting

To simplify the presentation we deal here with the case $s = 1$. The results extend to $s > 1$.

The locality of $R_{\mathbf{a}^k}$ can be more emphatically expressed in terms of two finite-dimensional matrices, which are both sections of the bi-infinite matrix A^k in (2.18). First we obtain the two finite-dimensional matrices. Consider

$$S_0^{\infty}\mathbf{f}^0 = S_{\{\mathbf{a}^k\}}^{\infty}\mathbf{f}^0 = \sum_{\alpha\in\mathbb{Z}} f_\alpha^0\phi_0(\cdot-\alpha) = \sum_{\alpha\in\mathbb{Z}} f_\alpha^k\phi_k(2^k\cdot-\alpha), \tag{5.1}$$

and its restriction to a unit interval. Due to the finite support of ϕ_0, there exists a finite set $I \subset \mathbb{Z}$, such that

$$S_{\{\mathbf{a}^k\}}^{\infty}\mathbf{f}^0\Big|_{[j,j+1]} = \sum_{\alpha-j\in I} f_\alpha^0\phi_0(\cdot-\alpha). \tag{5.2}$$

Thus the vector $\{f^0_\alpha : \alpha - j \in I\}$ completely determines the limit function in $[j, j+1]$. By the same reasoning, and since $\sigma(\phi_k) = \sigma(\phi_0)$, $k \in \mathbb{Z}_+$, we deduce, in view of (5.1), that the vector $\{f^k_\alpha : \alpha - j \in I\}$, with $\mathbf{f}^k = R_{\mathbf{a}^{k-1}} \cdots R_{\mathbf{a}^0} f^0$, determines the limit function in $[j, j+1]2^{-k}$. Again, by the linearity of $\{R_{\mathbf{a}^k} : k \in \mathbb{Z}_+\}$, there exists a linear map from $\{f^{k-1}_\alpha : \alpha \in I\}$ to $\{f^k_\alpha : \alpha \in I\}$, which is a square matrix of dimension $|I|$. We denote it by A^k_0. Similarly there is a linear transformation from $\{f^{k-1}_a : \alpha \in I\}$ to $\{f^k_\alpha : \alpha - 1 \in I\}$, denoted by A^k_1. Note that, by the uniformity of $R_{\mathbf{a}^k}$, A^k_ε maps the vector $\{f^{k-1}_\alpha : \alpha - j \in I\}$ to $\{f^k_\alpha : \alpha - j - \varepsilon \in I\}$, $\varepsilon = 0, 1$. It is easy to conclude from the definition of A^k_0, A^k_1 as linear operators, that the matrices A^k_0, A^k_1 are finite sections of the bi-infinite matrix A^k in (2.19), that is,

$$\begin{aligned} (A^k_0)_{\alpha\beta} &= a^k_{\alpha - 2\beta}, \quad \alpha, \beta \in I, \\ (A^k_1)_{\alpha\beta} &= a^k_{\alpha+1-2\beta}, \quad \alpha, \beta \in I. \end{aligned} \tag{5.3}$$

In the following we show how to get the value $(S_{\{\mathbf{a}^k\}} f^0)(x)$ for $x \in \mathbb{R}$ in terms of the matrices $\{A^k_0, A^k_1 : k \in \mathbb{Z}_+\}$. It is sufficient to consider the interval $[0, 1)$.

For $x \in [0, 1)$, we use the dyadic representation $x = \sum_{i=1}^\infty d_i 2^{-i}$, $d_i \in \{0, 1\}$, and obtain

$$\big(S^\infty_{\{\mathbf{a}^k\}} f^0\big)(x) = \lim_{k \to \infty} A^k_{d_{k+1}} A^{k-1}_{d_k} \cdots A^0_{d_1} \mathbf{f}^0_{[0,1]} \tag{5.4}$$

where $\mathbf{f}^0_{[0,1)} = \{f^0_\alpha : \alpha \in I\}$. Note that the finite product $A^k_{d_{k+1}} \cdots A^0_{d_1} \mathbf{f}^0_{[0,1)}$ is a vector which determines the limit function in an interval of the form $[j, j+1]2^{-k-1}$ containing x. Thus the convergence and smoothness of the limit function generated by $S_{\{\mathbf{a}^k\}}$ can be deduced from the set of finite matrices

$$\{A^k_0, A^k_1 : k \in \mathbb{Z}_+\} \tag{5.5}$$

and their infinite products of the form appearing in (5.4). In the stationary case there are only two matrices A_0, A_1, and all possible infinite products of them have to be considered (Micchelli and Prautzsch 1989).

5.2. Convergence and smoothness of univariate stationary schemes in terms of finite matrices

In the stationary case the value $(S^\infty_{\mathbf{a}} \mathbf{f}^0)(x)$ for $x = \sum_{j=1}^\infty d_j 2^{-j} \in [0, 1)$, $d_j \in \{0, 1\}$ is given by

$$(S^\infty_{\mathbf{a}} \mathbf{f}^0)(x) = \lim_{k \to \infty} A_{d_k} \cdots A_{d_1} \mathbf{f}^0_{[0,1)} \tag{5.6}$$

with $\mathbf{f}^0_{[0,1)} = \{f^0_\alpha : \alpha \in I\}$.

Note that $S_{\mathbf{a}}$ is contractive if and only if the joint spectral radius of A_0, A_1, $\rho_\infty(A_0, A_1)$, is less than 1, where

$$\rho_\infty(A_0, A_1) = \sup_{k \in \mathbb{Z}_+ \setminus 0} \left(\sup \left\{ \|A_{\varepsilon_k} A_{\varepsilon_{k-1}} \cdots A_{\varepsilon_1}\|_\infty : \varepsilon_i \in \{0,1\}, i = 1, \ldots, k \right\} \right)^{\frac{1}{k}}. \quad (5.7)$$

Thus the conditions for convergence and smoothness of a stationary scheme given in Section 4.2, which can be expressed as the contractivity of a related scheme, can be formulated in terms of the joint spectral radius of two finite matrices. (See, for instance, Daubechies and Lagarias (1992*b*).) It is easy to conclude that $\rho_\infty(A_0, A_1) \geq \max\{\rho(A_0), \rho(A_1)\}$, where $\rho(A)$ is the spectral radius of the matrix A. From this inequality and from the necessity of the contractivity condition, we obtain necessary conditions for convergence and smoothness (for the latter only in case of L_∞-stability), which are easy to check.

Such necessary conditions are important in the design of new schemes, in the sense that 'bad' schemes can easily be excluded. For example, if $a(z) = \frac{(1+z)^2}{2} b(z)$, and $S_{\mathbf{a}}$ is an interpolatory scheme, then $\rho(B_0) < 1$, and $\rho(B_1) < 1$ (with B_0 and B_1 the local matrix operators corresponding to $S_{\mathbf{b}}$) are necessary for $S_{\mathbf{a}}$ to be C^1.

Here we formulate an open problem: What are the conditions for the contractivity of $S_{\{\mathbf{a}^k\}}$ in terms of the matrices $\{A_0^k, A_1^k : k \in \mathbb{Z}_+\}$?

5.3. L_p-convergence and p-smoothness of univariate stationary schemes in terms of finite matrices

There is a vast literature (see, *e.g.*, Villemoes (1994), Jia (1995, 1999), Ron and Shen (2000), Han (1998), Han and Jia (1998) and Han (2001), and references therein) on the convergence in the L_p-norm of subdivision schemes, and on the p-smoothness of refinable functions. One central method of analysis is in terms of the p-norm joint spectral radius of two operators restricted to a finite-dimensional space.

Let A_0, A_1, be matrices of order $n \times n$. Their p-norm joint spectral radius is

$$\rho_p(A_0, A_1) = \sup_{k \in \mathbb{Z}_+ \setminus 0} \left(\left(\sum_{\varepsilon_1, \ldots, \varepsilon_k \in \{0,1\}} \|A_{\varepsilon_k} \cdots A_{\varepsilon_1}\|_p^p \right)^{\frac{1}{p}} \right)^{\frac{1}{k}}, \quad 1 \leq p < \infty.$$

For $\phi \in L_p(\mathbb{R})$ of compact support, the p-smoothness exponent is defined as

$$\nu_p(\phi) = \sup\{v \geq 0 : \|\Delta_h^n \phi\|_p \leq C h^v\}$$

for some constant $C > 0$ and for sufficiently large n, where $\Delta_h \phi = \phi - \phi(\cdot - h)$ and $\Delta_h^n = \Delta_h \Delta_h^{n-1}$.

Here we bring one result from Jia (1995), which is in some sense an extension of Theorem 4.9 in Section 4.2.

Theorem 5.1. Let **a** be a finitely supported mask such that $\sum_{i\in\mathbb{Z}} a_i = 2$. Let $\phi_{\mathbf{a}}$ be a nontrivial solution of the refinement equation

$$\phi_{\mathbf{a}} = \sum_{i\in\mathbb{Z}} a_i \phi_{\mathbf{a}}(2\cdot - i).$$

If there exists $C > 0$ such that $\|\Delta R_{\mathbf{a}}^n \delta\|_p^{1/n} \le C 2^{1/p-\mu}$ for $0 < \mu \le 1$ and $1 \le p \le \infty$, then $v_p(\phi_{\mathbf{a}}) = \mu$.

Since $\|\Delta R_{\mathbf{a}}^n \delta\|_p^{1/n} = \rho_p(A_0|_V, A_1|_V)$ with $V = \{\mathbf{u} \in \mathbb{R}^{|I|} : \sum_{i\in I} u_i = 0\}$ (Jia 1995), the condition of the above theorem can be formulated in terms of two finite-dimensional matrices, which are the restrictions of two operators to a finite-dimensional subspace. In Han (2001), an algorithm is presented for computing $v_2(\phi_{\mathbf{a}})$ efficiently, for $\phi_{\mathbf{a}}$ a multivariate refinable function corresponding to a dilation matrix M and a mask **a**, both with the same symmetries. For symmetric interpolatory masks there is also an algorithm for the computation of $\nu_\infty(\phi_{\mathbf{a}})$. The situation in the multivariate case is much more complex: there are $|\det M|$ operators, and the finite-dimensional univariate subspace to which these operators are restricted is quite complicated.

6. Extraordinary point analysis

For all the types of subdivision schemes that are defined over nets of arbitrary topology, as described in Section 3.5, the refined nets are regular nets, excluding a fixed number of extraordinary (irregular) points of valency $\neq 6$, in the case of triangular nets, and of valency $\neq 4$, in the case of quadrilateral nets. The smoothness analysis of subdivision schemes over nets of arbitrary topology is thus decomposed into two stages. First, the analysis over the regular part is completed, using the tools described in Sections 4 and 5. After verifying the smoothness over the regular part, we are left with a finite number of isolated points of unknown regularity. The regularity analysis at the extraordinary points has been studied by several authors, starting with the pioneering eigenvalue analysis work by Doo and Sabin (1978), through the works by Ball and Storry (1988, 1989), and completed by Prautzsch (1998), Reif (1995) and Zorin (2000). It is based on the observation that the regularity of the surface is known over a ring of patches Q^k encircling the extraordinary point, and there is a linear transformation T mapping the ring of patches Q^k onto a refined ring of patches, Q^{k+1}. Figure 6.1 displays a graphical description of three rings of patches around a vertex of valency five. The rings, each composed of 15 quadrilaterals, are self-similar, of reducing sizes.

The closure of the union of these rings defines an extraordinary patch covering a 'hole' in the regular part of the surface, and the smoothness of such a patch is completely characterized by the transformation T. In the following we present the key ingredients of the smoothness analysis of such patches and the main results.

Let us denote the basic limit function of the subdivision on a regular net by ϕ. The ring of patches Q^k may be expressed in terms of the control points P^k influencing this ring. Let $P^k = \{P_1^k, P_2^k, \ldots, P_N^k\} \subset \mathbb{R}^3$ be the control points generating Q^k, and let the transformation T be the square matrix such that $P^{k+1} = TP^k$.

Each patch in the ring $Q_\ell^k \in Q^k$ is a parametric patch, triangular or quadrilateral, which is a linear combination of translations of $\phi(2^k \cdot)$ multiplying control points $\{P_r^k\}_{r \in I_\ell} \subset P^k$. In other words,

$$Q^k = \bigcup Q_\ell^k, \tag{6.1}$$

where

$$Q_\ell^k = \left\{ q_\ell^k(u,v) \equiv \sum_{r \in I_\ell} P_r^k \phi(2^k u - i_r, 2^k v - j_r) \mid (u,v) \in \Omega \right\} \tag{6.2}$$

for appropriate $\{i_r, j_r\}_{r \in I_\ell}$. $\Omega = \{(u,v) \mid 0 \le u, v \le 1\}$ for quad-meshes and $\Omega = \{(u,v) \mid 0 \le u, v \ \wedge \ u + v \le 1\}$ for triangular meshes.

Since the regularity of ϕ is assumed to be already known, it is clear that the behaviour at the extraordinary vertex is completely characterized by the matrix T. It is important to note that the conditions for regularity at the extraordinary vertex do not require the knowledge of the explicit formula of ϕ. Using a proper ordering of the points P^k (Doo and Sabin 1978), the matrix T is a block-circulant matrix. The eigenvalue analysis of this matrix

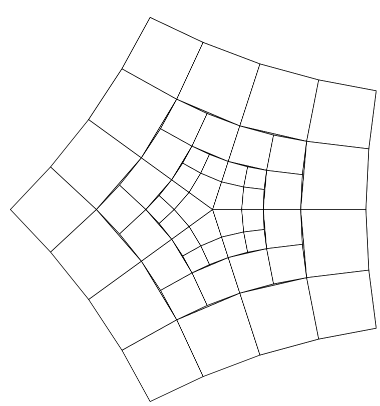

Figure 6.1. Three rings of patches

plays a crucial role in the smoothness analysis, as described in Doo and Sabin (1978), Reif (1995), Zorin (2000) and Prautzsch (1998). The results include necessary and sufficient conditions for geometric continuity, that is, existence of continuous limit normals at the extraordinary vertex and necessary and sufficient conditions for C^m-continuity at an extraordinary vertex – under some assumptions.

Let the eigenvalues $\lambda_0, \ldots, \lambda_{N-1}$ of T be ordered by modulus, that is,

$$|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_{N-1}|, \tag{6.3}$$

and let $V_0, V_1, \ldots, V_{N-1} \in \mathbb{R}^N$ denote the corresponding generalized real eigenvectors, assuming they exist.

As first shown in Doo and Sabin (1978), a necessary condition for the continuity of the normal at an extraordinary point is

$$\lambda_0 = 1 > |\lambda_1| = |\lambda_2| > |\lambda_3|, \quad V_0 = \{1, 1, \ldots, 1\}. \tag{6.4}$$

Assuming (6.4) holds, let us consider the particular initial data vector

$$P^0 = \{P_1^0, P_2^0, \ldots, P_N^0\}$$

with

$$P_j^0 = (V_{1,j}, V_{2,j}, 0)^t, \tag{6.5}$$

and let us examine the corresponding rings of patches defined by (6.2).

Injectivity and regularity assumption. We assume that each mapping q_ℓ^0 in (6.2) is regular and injective, and that

$$\bigcap_\ell \mathrm{int}\{Q_\ell^0\} = \emptyset. \tag{6.6}$$

In Reif (1995), the collection of mappings $\{q_\ell^0\}$ is termed the 'characteristic map' and the above assumption is thus referred to as the regularity and injectivity of the characteristic map. The importance of this map is that it defines the natural parametric domain for analysing the smoothness of the surface at the extraordinary vertex. For a discussion and analysis of the characteristic map see Peters and Reif (1998). Under the above assumption sufficient conditions for C^1 regularity are presented in the following result from Reif (1995).

Theorem 6.1. Let (6.3) hold with $\lambda_1 = \lambda_2$ being a real eigenvalue of T with geometric multiplicity 2, and let the characteristic map be regular and injective. Then the limit surface of the subdivision is a regular C^1-manifold in a neighbourhood of the extraordinary vertex for almost any initial data.

The necessary and sufficient conditions for C^m-continuity at an extraordinary vertex were derived independently by Prautzsch (1998) and Zorin (2000). These results are equivalent to the polynomial reproduction result for uniform stationary C^m-schemes on regular meshes.

Theorem 6.2. **(C^m-conditions)** Let the conditions of Theorem 6.1 hold. Then the limit surface of the subdivision is a regular C^m manifold in a neighbourhood of the extraordinary vertex for almost any initial data, if and only if the following condition holds.

For any eigenvalue λ of T satisfying $|\lambda| > \lambda_1^m$:

(a) $|\lambda| = \lambda_1^i$ for some integer $0 \leq i \leq m$;

(b) for the initial data vector $P^0 = \{P_1^0, P_2^0, \ldots, P_N^0\}$ with

$$P_j^0 = (V_{1,j}, V_{2,j}, V_j)^t \in \mathbb{R}^3, \tag{6.7}$$

and V an eigenvector corresponding to λ, all the patches Q_ℓ^0 lie on a polynomial surface $z = p(x, y)$ in $\mathbb{R}^3$, where p is a homogeneous polynomial of total degree i.

Theorem 6.2 does not give explicit constructive conditions that can help us to build a C^m-scheme. The translation of the conditions in Theorem 6.2 into algebraic conditions on the mask coefficients is rather complicated, and even in the C^2 case is not fully resolved. The partial results in this direction include the construction of schemes with bounded curvatures, in Loop (2001), and the special patch construction by Prautzsch and Umlauf (1998). For some applications it is enough to have curvature integrability of the subdivision surface. Reif and Schröder (2000) show that the Catmull–Clark and Loop schemes (among many others) have square integrable principal curvatures.

7. Limit values and approximation order

In this section we discuss two practical issues in the implementation of subdivision algorithms in geometric modelling. One issue is the computation of limit values and limit derivatives of the subdivision process at the dyadic points of any refinement level. The other important issue, though not yet widely appreciated, is how to actually attain the optimal approximation order for a given scheme: in other words, how to choose the initial control points so that the limit curve/surface will approximate a desired curve/surface with the highest possible approximation power.

7.1. Limit values and derivative values

We consider here only the stationary case, namely when $\mathbf{a}^k = \mathbf{a}$, $k \in \mathbb{Z}_+$, and assume that the basic limit function $\phi \equiv \phi_{\mathbf{a}}$ is C^m. The support of ϕ is contained in the convex hull of the support of the mask, $\sigma(\mathbf{a})$, by (2.13). Furthermore, by (2.10) we can express the limit function of $S_{\mathbf{a}}$ as

$$f \equiv S_{\mathbf{a}}^\infty \mathbf{f}^0 = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha^0 \phi(\cdot - \alpha). \tag{7.1}$$

Thus, the limit values at the integer points $\beta \in \mathbb{Z}^s$ are given by

$$f(\beta) = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha^0 \phi(\beta - \alpha). \tag{7.2}$$

By (7.2), knowledge of the values of ϕ at integer points gives one the possibility of computing the limit values of the subdivision process on the integer grid $\mathbb{Z}^s$, using only the initial control points $\mathbf{f}^0$. Similarly, the limit values on the dyadic grid $2^{-k}\mathbb{Z}^s$ are defined by the control points $\mathbf{f}^k$ at level k. In the same way we note that the values of a derivative of f at the integers are linear combinations of the values of the same derivative of ϕ at the integers. The vector of values of ϕ, or of one of its derivatives, at the integer points may each be computed as the eigenvector of a finite matrix.

To see this we recall that ϕ satisfies the refinement equation (2.15), and thus

$$\partial_\lambda \phi = 2^{|\lambda|} \sum_\alpha a_\alpha \partial_\lambda \phi(2 \cdot - \alpha), \tag{7.3}$$

where $\lambda \in \mathbb{Z}_+^s$, $|\lambda| = \sum_{i=1}^s \lambda_i \le m$. At integer points $\beta \in \mathbb{Z}^s$ we have the linear relations

$$\partial_\lambda \phi(\beta) = 2^{|\lambda|} \sum_\alpha a_\alpha \partial_\lambda \phi(2\beta - \alpha) = 2^{|\lambda|} \sum_\gamma a_{2\beta-\gamma} \partial_\lambda \phi(\gamma). \tag{7.4}$$

Now, since ϕ is of compact support, there is only a finite number N_ϕ of grid points where ϕ is nonzero. Let $\Omega \equiv \mathbb{Z}^s \bigcap \sigma(\phi)$; then $N_\phi = \#\Omega$. The system of equations (7.4), with $\beta \in \Omega$, is a square $N_\phi \times N_\phi$ eigensystem for the values $\{\partial_\lambda \phi(\beta)\}_{\beta\in\Omega}$, and it has a unique solution if we add the side conditions

$$\sum_{-\beta\in\Omega} \beta^\lambda \partial_\lambda \phi(-\beta) = \lambda!, \quad \sum_{-\beta\in\Omega} \beta^\mu \partial_\lambda \phi(-\beta) = 0, \quad \mu \ne \lambda, \quad |\mu| = |\lambda|. \tag{7.5}$$

These side conditions, in view of (7.2), guarantee that the $|\lambda|$ order derivatives of $S_{\mathbf{a}}^\infty x^\mu|_{\mathbb{Z}^s}$ are correctly obtained, for $|\mu| = |\lambda|$. For example, in the univariate case, the vector of values $\{\phi(\beta)\}$ is an eigenvector of the matrix U with elements $U_{i,j} = a_{2i-j}$, corresponding to the eigenvalue 1, and with the normalization $\sum \phi(\beta) = 1$. The vector of values $\{\phi'(\beta)\}$ is an eigenvector of U with eigenvalue 2. Implementing this, the rule for computing the limit derivatives of a curve defined by the 4-point scheme (3.18) turns out to be (Dyn *et al.* 1987):

$$f'(2^{-k}i) = \frac{2^k}{1-4w}\left[\frac{1}{2}(f_{i+1}^k - f_{i-1}^k) - w(f_{i+2}^k - f_{i-2}^k)\right]. \tag{7.6}$$

The method for computing limit values is actually applied to non-interpolatory subdivision surfaces, so that at all refinement levels the rendered points are on the limit surface. The shading of the surface at each level is done with normals which are the actual normals of the limit surface. A detailed example of computing limit normals at regular points and at extraordinary points for the case of the butterfly scheme is given in Shenkman (1996) and Dyn, Levin and Shenkman (1999*b*).

7.2. Attaining the optimal approximation order

The term *approximation order* of a subdivision scheme $S_{\mathbf{a}}$ refers to the rate by which the limit functions generated by $S_{\mathbf{a}}$, from initial data sampled from a sufficiently smooth function f, get closer to f: in other words, the largest exponent r such that

$$\|f - S_{\mathbf{a}}^{\infty} f|_{h\mathbb{Z}^s}\|_\infty \leq ch^r.$$

Yet this order may be improved (for non-interpolatory schemes) by replacing the initial data $f|_{h\mathbb{Z}^s}$ by $Qf|_{h\mathbb{Z}^s}$ with Q a Toeplitz operator of finite support. Our aim is to find the operator Q that yields the largest approximation rate.

Let us start with an example.

Example 9. Let us consider the case of univariate cubic B-splines with integer knots. It is known that the integer shifts of this cubic B-spline, B_3, span π_3, and this implies that the space generated by the integer shifts of the cubic B-spline has potential approximation order 4. If $f \in C^4(\mathbb{R})$, then the use of function values as control points gives a second-order approximation, by the corresponding subdivision scheme

$$\left| f(x) - \sum_{j\in\mathbb{Z}} f(jh) B_3\left(\frac{x}{h} - j\right)\right| \leq c_2 h^2. \tag{7.7}$$

However, using as control points the values

$$\tilde{f}_j = (Q_h f)(jh) \equiv -\frac{1}{6} f((j-1)h) + \frac{4}{3} f(jh) - \frac{1}{6} f((j+1)h), \tag{7.8}$$

we get the optimal fourth-order approximation:

$$\left| f(x) - \sum_{j\in\mathbb{Z}} \tilde{f}_j B_3\left(\frac{x}{h} - j\right)\right| \leq c_4 h^4. \tag{7.9}$$

This special choice of Q_h is made so that the approximation scheme in (7.9) reproduces all polynomials in $\pi_3(\mathbb{R})$, namely, $\sum (Q_h p)(jh) B_3(\frac{x}{h} - j) = p(x)$ for any $p \in \pi_3(\mathbb{R})$. Therefore, to approximate a curve $c(t)$ by a cubic spline subdivision, given a sequence of points $\{P_j\}$ ordered on it, then it is better to start the subdivision process with the control points

$$\widetilde{P}_j = -\frac{1}{6} P_{j-1} + \frac{4}{3} P_j - \frac{1}{6} P_{j+1}. \tag{7.10}$$

The above idea is extended for general subdivision schemes in A. Levin (1999c).

For a given uniform stationary scheme $S_{\mathbf{a}}$ we identify the maximal m such that $\pi_m(\mathbb{R}^s)$ is invariant under $S_{\mathbf{a}}$ in the sense that $S_{\mathbf{a}}^\infty p|_{\mathbb{Z}^s} \in \pi_m(\mathbb{R}^s)$ for any $p \in \pi_m(\mathbb{R}^s)$. Then, the potential approximation order is $m+1$. To achieve this approximation power we look for a Toeplitz operator Q, of minimal support Σ, of the form

$$(Q\mathbf{f}^0)_\alpha = \sum_{\sigma\in\Sigma} q_\sigma f^0_{\alpha-\sigma}, \tag{7.11}$$

such that

$$S_{\mathbf{a}}^\infty Q(p|_{\mathbb{Z}^s}) = p, \quad \forall p \in \pi_m(\mathbb{R}^s). \tag{7.12}$$

In other words, Q is the inverse of $S_{\mathbf{a}}^\infty$ on $\pi_m(\mathbb{R}^s)$. If Q exists then it commutes with $S_{\mathbf{a}}^\infty$ on π_m. Therefore, we look for Q such that $QS_{\mathbf{a}}^\infty p|_{\mathbb{Z}^s} = p, \quad \forall p \in \pi_m(\mathbb{R}^s)$. Using the results of Section 7.1 we can define the polynomials

$$r_\gamma \equiv S_{\mathbf{a}}^\infty\{x^\gamma|_{\mathbb{Z}^s}\} = \sum_{\alpha\in\mathbb{Z}^s} \phi(\alpha)(\cdot-\alpha)^\gamma, \quad \gamma\in\mathbb{Z}^s, \quad |\gamma| \le m,$$

which constitute a basis of π_m. Now we look for an operator Q such that on π_m it is the inverse of $S_{\mathbf{a}}^\infty$, namely,

$$Qr_\gamma = x^\gamma, \quad \gamma\in\mathbb{Z}^s, \quad |\gamma| \le m. \tag{7.13}$$

This can be formulated as a system of linear equations in the finite-dimensional space π_m,

$$\sum_{\sigma\in\Sigma} q_\sigma r_\gamma(x-\sigma) = x^\gamma, \quad |\gamma| \le m. \tag{7.14}$$

In the above example of the cubic B-splines, the operator Q may also be chosen to be $Qf = f - \frac{1}{6}f''$, or to be the difference operator given in (7.8). The two options act in the same way on π_3, yet, for the purpose of applying Q on the given data points we need the discrete form (7.8). For further examples and applications see A. Levin (1999c).

Acknowledgements

The authors wish to thank Nurit Alkalai and Adi Levin for providing the figures, and to Bin Han and Peter Schröder for their help with the references. This work was supported in part by a grant from the Israeli Academy of Sciences (Center of Excellence program), and by the European Union research project 'Multiresolution in Geometric Modelling (MINGLE)' under grant HPRN-CT-1999-00117.

REFERENCES

A. A. Ball and D. J. T. Storry (1988), 'Conditions for tangent plane continuity over recursively generated B-spline surfaces', *ACM Trans. Graphics* **7**, 83–102.

A. A. Ball and D. J. T. Storry (1989), 'Design of an n-sided surface patch', *Computer Aided Geometric Design* **6**, 111–120.

C. de Boor (1987), 'Cutting corners always works', *Computer Aided Geometric Design* **4**, 125–131.

C. de Boor, K. Höllig and S. Riemenschneider (1993), *Box Splines*, Vol. 98 of *Applied Mathematical Sciences*, Springer.

E. Catmull and J. Clark (1978), 'Recursively generated B-spline surfaces on arbitrary topological meshes', *Computer Aided Design* **10**, 350–355.

A. S. Cavaretta, W. Dahmen and C. A. Micchelli (1991), *Stationary Subdivision*, Vol. 453 of *Memoirs of AMS*, American Mathematical Society.

G. M. Chaikin (1974), 'An algorithm for high speed curve generation', *Computer Graphics and Image Processing* **3**, 346–349.

A. Cohen and Conze (1992), 'Regularité des bases d'ondelettes et mesures ergodiques', *Rev. Math. Iberoamer* **8**, 351–366.

A. Cohen and N. Dyn (1996), 'Nonstationary subdivision schemes and multiresolution analysis', *SIAM J. Math. Anal.* **26**, 1745–1769.

A. Cohen, I. Daubechies and G. Plonka (1997), 'Regularity of refinable function vectors', *J. Fourier Anal. Appl.* **3**, 295–324.

A. Cohen, N. Dyn and D. Levin (1996), Stability and inter-dependence of matrix subdivision schemes, in *Advanced Topics in Multivariate Approximation* (F. Fontanella, K. Jetter and P. J. Laurent, eds), World Scientific, pp. 33–45.

A. Cohen, N. Dyn and B. Matei (2001), Quasilinear subdivision schemes with applications to ENO interpolation. Submitted.

E. Cohen, T. Lyche and R. F. Riesenfeld (1980), 'Discrete B-splines and subdivision techniques in computer-aided geometric design and computer graphics', *Computer Graphics and Image Processing* **14**, 87–111.

W. Dahmen and C. A. Micchelli (1984), 'Subdivision algorithms for the generation of box spline surfaces', *Computer Aided Geometric Design* **1**, 115–129.

W. Dahmen and C. A. Micchelli (1997), 'Biorthogonal wavelet expansion', *Constr. Approx.* **13**, 294–328.

I. Daubechies (1992), *Ten Lectures on Wavelets*, SIAM, Philadelphia.

I. Daubechies and J. C. Lagarias (1992*a*), 'Two-scale difference equations, I: Existence and global regularity of solutions', *SIAM J. Math. Anal.* **22**, 1388–1410.

I. Daubechies and J. C. Lagarias (1992*b*), 'Two-scale difference equations, II: Local regularity, infinite products of matrices and fractals', *SIAM J. Math. Anal.* **23**, 1031–1079.

I. Daubechies, I. Guskov and W. Sweldens (1999), 'Regularity of irregular subdivision', *Constr. Approx.* **15**, 381–426.

S. Dekel and N. Dyn (2001), 'Polyscale subdivision schemes and refinability', *Appl. Comput. Harm. Anal.* To appear.

G. Derfel, N. Dyn and D. Levin (1995), 'Generalized refinement equations and subdivision processes', *J. Approx. Theory* **80**, 272–297.

T. DeRose, M. Kass and T. Truong (1998), Subdivision surfaces in character animation, in *Proc. SIGGRAPH 98*, Annual Conference Series, ACM–SIGGRAPH, pp. 85–94.

G. Deslauriers and S. Dubuc (1989), 'Symmetric iterative interpolation', *Constr. Approx.* **5**, 49–68.

D. Donoho and V. Stodden (2001), Multiplicative multiresolution analysis for lie-group valued data indexed by Euclidean parameter. In preparation.

D. Donoho, N. Dyn, D. Levin and T. Yu (2000), 'Smooth multiwavelet duals of Alpert bases by moment-interpolating refinement', *Appl. Comput. Harm. Anal.* **9**, 166–203.

D. Doo and M. Sabin (1978), 'Behaviour of recursive division surface near extraordinary points', *Computer Aided Design* **10**, 356–360.

S. Dubuc (1986), 'Interpolation through an iterative scheme', *J. Math. Anal. Appl.* **114**, 185–204.

N. Dyn (1992), Subdivision schemes in computer aided geometric design, in *Advances in Numerical Analysis II: Subdivision Algorithms and Radial Functions* (W. A. Light, ed.), Oxford University Press, pp. 36–104.

N. Dyn and E. Farkhi (2000), 'Spline subdivision schemes for convex compact sets', *J. Comput. Appl. Math.* **119**, 133–144.

N. Dyn and E. Farkhi (2001*a*), Convexification rates in Minkowski averaging processes. Submitted.

N. Dyn and E. Farkhi (2001*b*), Spline subdivision schemes for compact sets with metric averages, in *Trends in Approximation Theory* (T. L. K. Kopotun and M. Neamtu, eds), Vanderbilt University Press, pp. 93–102.

N. Dyn and D. Levin (1990), Interpolatory subdivision schemes for the generation of curves and surfaces, in *Multivariate Approximation and Interpolation* (W. Haussmann and K. Jetter, eds), Birkhäuser, Basel, pp. 91–106.

N. Dyn and D. Levin (1992), Stationary and non-stationary binary subdivision schemes, in *Mathematical Methods in Computer Aided Geometric Design II* (T. Lyche, and L. L. Schumaker, eds), Academic Press, pp. 209–216.

N. Dyn and D. Levin (1995), 'Analysis of asymptotically equivalent binary subdivision schemes', *J. Math. Anal. Appl.* **193**, 594–621.

N. Dyn and D. Levin (1999), Analysis of Hermite-interpolatory subdivision schemes, in *CRM Proceedings and Lecture Notes*, Vol. 18, Centre de Recherches Mathématiques, pp. 105–113.

N. Dyn and D. Levin (2002), Matrix subdivision: Analysis by factorization, in *Approximation Theory: A Volume Dedicated to Blagovest Sendov* (B. Bojanov, ed.), Darba, Sofia, pp 187–211.

N. Dyn and A. Ron (1995), 'Multiresolution analysis by infinitely differentiable compactly supported functions', *Appl. Comput. Harm. Anal.* **2**, 15–20.

N. Dyn, J. A. Gregory and D. Levin (1987), 'A four-point interpolatory subdivision scheme for curve design', *Computer Aided Geometric Design* **4**, 257–268.

N. Dyn, J. A. Gregory and D. Levin (1990*a*), 'A butterfly subdivision scheme for surface interpolation with tension control', *ACM Trans. Graphics* **9**, 160–169.

N. Dyn, J. A. Gregory and D. Levin (1991), 'Analysis of uniform binary subdivision schemes for curve design', *Constr. Approx.* **7**, 127–147.

N. Dyn, J. A. Gregory and D. Levin (1995), Piecewise uniform subdivision schemes, in *Mathematical Methods for Curves and Surfaces* (M. Dahlen, T. Lyche and L. L. Schumaker, eds), Vanderbilt University Press, Nashville, pp. 111–120.

N. Dyn, S. Hed and D. Levin (1993), Subdivision schemes for surface interpolation, in *Workshop on Computational Geometry* (A. Conte *et al.*, eds), World Scientific, pp. 97–118.

N. Dyn, F. Kuijt, D. Levin and R. van Damme (1999*a*), 'Convexity preservation of the four-point interpolatory subdivision scheme', *Computer Aided Geometric Design* **16**, 789–792.

N. Dyn, D. Levin and D. Liu (1992), 'Interpolatory convexity preserving subdivision schemes for curves and surfaces', *Computer Aided Design* **24**, 211–216.

N. Dyn, D. Levin and A. Luzzatto (2001*a*), Non-stationary interpolatory subdivision schemes reproducing spaces of exponential polynomials. Submitted.

N. Dyn, D. Levin and C. A. Micchelli (1990*b*), 'Using parameters to increase smoothness of curves and surfaces generated by subdivision', *Computer Aided Geometric Design* **7**, 129–140.

N. Dyn, D. Levin and P. Shenkman (1999*b*), 'Normals of the butterfly scheme surfaces and their applications', *J. Comput. Appl. Math.* **102**, 157–180.

N. Dyn, D. Levin and J. Simoens (2001*b*), Face value subdivision schemes on triangulations by repeated averaging. Preprint.

J. Gregory (1991), An introduction to bivariate uniform subdivision, in *Numerical Analysis 1991* (D. Griffiths and G. Watson, eds), Pitman Research Notes in Mathematics, Longman Scientific and Technical, pp. 103–117.

J. A. Gregory and R. Qu (1996), 'Non-uniform corner cutting', *Computer Aided Geometric Design* **13**, 763–772.

I. Guskov (1998), Multivariate subdivision schemes and divided differences. Technical report, Princeton University.

M. Halstead, M. Kass and T. DeRose (1993), Efficient, fair interpolation using Catmull–Clark surfaces, in *Proc. SIGGRAPH 93*, Annual Conference Series, ACM–SIGGRAPH, pp. 35–44.

B. Han (1998), 'Symmetric orthonormal scaling functions and wavelets with dilation factor 4', *Adv. Comput. Math.* **8**, 221–247.

B. Han (2001), Computing the smoothness exponent of a symmetric multivariate refinable function. Preprint.

B. Han and R. Q. Jia (1998), 'Multivariate refinement equations and convergence of subdivision schemes', *SIAM J. Math. Anal.* **29**, 1177–1199.

S. Hed (1990), Analysis of subdivision schemes for surfaces. Master's thesis, Tel Aviv University.

H. Hoppe, T. DeRose, T. Duchamp, M. Halstead, H. Jin, J. McDonald, J. Schweitzer and W. Stuetzle (1994), 'Piecewise smooth surface reconstruction', *Computer Graphics* **28**, 295–302.

R. Q. Jia (1995), 'Subdivision schemes in l_p spaces', *Adv. Comput. Math.* **3**, 309–341.

R. Q. Jia (1996), The subdivision and transition operators associated with a refinement equation, in *Advanced Topics in Multivariate Approximation* (K. J. F. Fontanella and L. Schumaker, eds), World Scientific, pp. 1–13.

R. Q. Jia (1999), 'Characterization of smoothness of multivariate refinable functions in Sobolev spaces', *Trans. Amer. Math. Soc.* **351**, 4089–4112.

R. Q. Jia and S. Zhang (1999), 'Spectral properties of the transition operator associated to a multivariate refinement equation', *Lin. Alg. Appl.* **292**, 155–178.

Y. Kasas (1990), A subdivision-based algorithm for surface/surface intersection. Master's thesis, Tel Aviv University.

L. Kobbelt (1996*a*), 'Interpolatory subdivision on open quadrilateral nets with arbitrary topology', *Computer Graphics Forum* **15**, 409–420.

L. Kobbelt (1996*b*), 'A variational approach to subdivision', *Computer Aided Geometric Design* **13**, 743–761.

L. Kobbelt, T. Hesse, H. Prautzsch and K. Schweizerhof (1996), 'Interpolatory subdivision on open quadrilateral nets with arbitrary topology', *Computer Graphics Forum* **15**, 409–420. *Eurographics '96* issue.

F. Kuijt and R. van Damme (1998), 'Convexity preserving interpolatory subdivision schemes', *Constr. Approx.* **14**, 609–630.

F. Kuijt and R. van Damme (1999), 'Monotonicity preserving interpolatory subdivision schemes', *J. Comput. Appl. Math.* **101**, 203–229.

F. Kuijt and R. van Damme (2002), 'Shape preserving interpolatory subdivision schemes for nonuniform data', *J. Approx. Theory.* To appear.

O. Labkovsky (1996), The extended butterfly interpolatory subdivision scheme for the generation of C^2 surfaces. Master's thesis, Tel Aviv University.

A. Levin (1999*a*), Analysis of quasi-uniform subdivision schemes. In preparation.

A. Levin (1999*b*), 'Combined subdivision schemes for the design of surfaces satisfying boundary conditions', *Computer Aided Geometric Design* **16**, 345–354.

A. Levin (1999*c*), Combined subdivision schemes with applications to surface design. PhD thesis, Tel Aviv University.

A. Levin (1999*d*), Interpolating nets of curves by smooth subdivision surfaces, in *Proc. SIGGRAPH 99*, Annual Conference Series, ACM–SIGGRAPH, pp. 57–64.

D. Levin (1999*e*), 'Using Laurent polynomial representation for the analysis of non-uniform binary subdivision schemes', *Adv. Comput. Math.* **11**, 41–54.

C. Loop (1987), Smooth spline surfaces based on triangles. Master's thesis, University of Utah, Department of Mathematics.

C. Loop (2001), Triangle mesh subdivision with bounded curvature and the convex hull property. Technical Report MSR-TR-2001-24, Microsoft Research.

S. Mallat (1989), 'Theory for multiresolution signal decomposition: The wavelet representation', *IEEE Trans. Pattern Anal. Mach. Intel.* **11**, 674–693.

J. L. Merrien (1992), 'A family of Hermite interpolants by bisection algorithms', *Numer. Alg.* **2**, 187–200.

C. A. Micchelli and H. Prautzsch (1989), 'Uniform refinement of curves', *Lin. Alg. Appl.* **114/115**, 841–870.

C. A. Micchelli and T. Sauer (1998), 'On vector subdivision', *Math. Z.* **229**, 621–674.

G. Morin, J. Warren and H. Weimer (2001), 'A subdivision scheme for surfaces of revolution', *Computer Aided Geometric Design* **18**, 483–502.

A. H. Nasri (1997*a*), 'Curve interpolation in recursively generated B-spline surfaces over arbitrary topology', *Computer Aided Geometric Design* **14**, 13–30.

A. H. Nasri (1997*b*), Interpolation of open curves by recursive subdivision surface, in *The Mathematics of Surfaces VII* (T. Goodman and R. Martin, eds), Information Geometers, pp. 173–188.

J. Peters and U. Reif (1997), 'The simplest subdivision scheme for smoothing polyhedra', *ACM Trans. Graphics* **16**, 420–431.

J. Peters and U. Reif (1998), 'Analysis of algorithms generating B-spline subdivision', *SIAM J. Numer. Anal.* **35**, 728–748.

G. Plonka (1997), Approximation order provided by refinable function vectors, *Constr. Approx.* **13** 221–244.

H. Prautzsch (1998), 'Smoothness of subdivision surfaces at extraordinary points', *Adv. Comput. Math.* **9**, 377–389.

H. Prautzsch and G. Umlauf (1998), Improved triangular subdivision schemes, in *Computer Graphics International 1998* (F. E. Wolter and N. M. Patrikalakis, eds), IEEE Computer Society, pp. 626–632.

G. de Rahm (1956), 'Sur une courbe plane', *J. Math. Pures Appl.* **35**, 25–42.

U. Reif (1995), 'A unified approach to subdivision algorithms near extraordinary points', *Computer Aided Geometric Design* **12**, 153–174.

U. Reif and P. Schröder (2000), 'Curvature integrability of subdivision surfaces', *Adv. Comput. Math.* **12**, 1–18.

D. Riemenschneider and Z. Shen (1997), 'Multidimensional interpolatory subdivision schemes', *SIAM J. Numer. Anal.* **34**, 2357–2381.

R. F. Riesenfeld (1975), 'On Chaikin's algorithm', *Computer Graphics and Image Processing* **4**, 304–310.

O. Rioul (1992), 'Simple regularity criteria for subdivision schemes', *SIAM J. Math. Anal.* **23**, 1544–1576.

A. Ron and Z. Shen (2000), 'The Sobolev regularity of refinable functions', *J. Approx. Theory* **106**, 185–225.

V. A. Rvachev (1990), 'Compactly supported solutions of functional–differential equations and their applications', *Russian Math. Surveys* **45**, 87–120.

P. Schröder (2001), Subdivision, multiresolution and the construction of scalable algorithms in computer graphics, in *Multivariate Approximation and Applications*, Cambridge University Press, pp. 213–251.

L. L. Schumaker (1980), *Spline Functions: Basic Theory*, Wiley-Interscience.

P. Shenkman (1996), Computing normals and offsets of curves and surfaces generated by subdivision schemes. Master's thesis, Tel Aviv university.

L. F. Villemoes (1994), 'Wavelet analysis of refinement equations', *SIAM J. Math. Anal.* **25**, 1433–1460.

J. Warren (1995*a*), Binary subdivision schemes for functions over irregular knot sequences, in *Mathematical Methods in CAGD III* (M. Dahlen, T. Lyche and L. L. Schumaker, eds), Vanderbilt University Press, Nashville.

J. Warren (1995*b*), *Subdivision Methods for Geometric Design*, Rice University.

D. Zorin (2000), 'Smoothness of subdivision on irregular meshes', *Constr. Approx.* **16**, 359–397.

D. Zorin and P. Schröder (2000), *Subdivision for Modeling and Animation*, Course Notes, ACM–SIGGRAPH.

D. Zorin, H. Biermann and A. Levin (2000), Piecewise smooth subdivision surfaces with normal control, in *Proc. SIGGRAPH 2000*, Annual Conference Series, pp. 113–120.

D. Zorin, P. Schröder and W. Sweldens (1996), Interpolating subdivision for meshes with arbitrary topology, in *Proc. SIGGRAPH 96*, Annual Conference Series, ACM–SIGGRAPH, pp. 189–192.

D. Zorin, P. Schröder and W. Sweldens (1997), Interactive multiresolution mesh editing, in *Proc. SIGGRAPH 97*, Annual Conference Series, ACM–SIGGRAPH, pp. 259–268.